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Original Research

Jump-Diffusion Model for Excess Volatility in Asset Prices: Generalized Langevin Equation Approach

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ABSTRACT

The excess volatility puzzle refers to the observation of returns that cannot be explained only by fundamentals, and this research attempts to explain such volatilities using the concept of endogenous jumps and modelling them based on the generalized Langevin equation. Based on stylized facts, price behaviour in financial markets is not simply a continuous process, but rather jumps are observed in asset prices that may be exogenous or endogenous. It is claimed that the source of exogenous jumps is news, and the source of endogenous jumps is internal interactions between the agents. The goal is to extract these endogenous jumps as a function of the state variable and time. For this purpose, the generalized Langevin equation is introduced and it is shown that the parameters of this model can be extracted based on the Kramers-Moyal coefficients. The results of self-consistency tests to evaluate the accuracy of the Kramers-Moyal method in extracting the generalized Langevin equation show that this method has good accuracy. In a practical application of the aforementioned method, Ethereum cryptocurrency price data was used between October 2017 and February 2024 with a sampling rate of one minute. By simulating the extracted dynamics, the probability distribution of the first time passage of this cryptocurrency from a specific level was calculated, and an examination of the price behavior of this asset shows that the aforementioned distribution was extracted with good accuracy. The potential function, which is calculated from the first KM coefficient, will be a quadratic parabola for the studied process, and as a result, we have a stable equilibrium at the zero point. Also using the extracted dynamics we show that this model has good out-of-sample prediction ability.

1 Introduction

Do all observed jumps in asset prices have an external source or do some of these jumps arise from internal mechanisms? This is a question that has been raised as an open problem by Cutler et al. [1] and has been the subject of extensive research by economists and also by physicists interested in the financial field. According to Cutler's paper, observing stock price jumps when no news related to the stock was published at the time of these jumps requires us to look for factors other than news to explain the

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aforementioned jumps. In other words, large price fluctuations cannot be explained solely by relying on observed changes in fundamentals, and the amplitude of these fluctuations is greater than that predicted by fundamental models, therefore we must look for other factors to explain the additional fluctuations.The problem of the inconsistency of the volatility amplitude with the observed changes in fundamental factors, has been studied in more detail by Julin et al.[2]. based on the results of aforermentioned research, neither the news related to a single share nor the macroeconomic news alone can explain the frequency and amplitude of fluctuations observed in the returns of a single share. According to the results of this research, about 90% of the large fluctuations observed in the financial markets cannot be explained through the news seen in Bloomberg, Reuters or other main sources of information for investors, and in fact only about 5 to 10% of the fluctuations that are larger than 4σ can be explained based on news. Therefore, our problem in this research has two parts: 1- The first part which seeks to answer the question of what is the mechanism behind the observed jumps in asset prices? Do these jumps have an external stimulus or do mechanisms from within the system create them? 2- The second part, which must model and regenerate the jumps observed in real data.

If we accept the exogeneity of the jumps based on the standard theories in the economics, then statistically we reach an obvious contradiction, because the real observations confirm the existence of fat tails in asset returns [3,4]. The presence of the fat tails in the distribution of related data, indicates that fluctuations with a large amplitude have a high frequency and have been repeated in a large number. Therefore, if we accept that the fluctuations with large amplitude caused only by news, then we will come to a contradiction about the frequencies, because the frequency of the news is far less than the frequency observed in jumps. The standard view in economics considers these fluctuations to be caused by external shocks [5]. But in the modern view, that is, complex systems theory, system dynamics are considered based on the combination of the effect of the internal interactions of the agents (endogenous shock) and the effect of external shocks (news). The internal interactions of the agents in the financial markets is mainly caused by narratives formed in social networks, herding behavior and imitation of other traders[6] or by the fear and greed of traders[7] which ultimately leads to over-reactions or under-reactions observed in the market.The purpose of the reconstruction of the stochastic process that we do in this research is to extract the dynamics of asset prices in the short term between two macro shocks and to reconstruct endogenous jumps that cannot be explained based on external shocks. As we will see below, jump-diffusion models allow us to have large and of course endogenous jumps even in the absence of macro news, and this is something that is more consistent with the reality of financial markets, because if the price behaviour in the financial markets are formed based on the hypothesis of rational expectations, then according to the claim of this theory - about the rapid adjustment of prices and the rapid formation of a new equilibrium after the occurrence of a shock - prices should have exhibited perturbative behaviour (small fluctuations around the new equilibrium) until the next external shock occurred. But Cutler, et al.[1]; Joulin, et al.[2] and Farmer[8] show that, in the real world we see jumps in asset prices even in the absence of news affecting stocks. By adding endogenous shocks to the problem, we can propose the idea that "the market makes its own news", which means that even in the absence of external shocks and due to the internal interaction of the agents, the market can have non-perturbative dynamics. Our goal in general is to extract the macroscopic behaviour of the system (and here the price as the macroscopic behavior of the market) from the observations. We note that extracting macroscopic behavior means that although the price is the result of the microscopic interaction of individual traders in the market, but here we model the price behavior only based on the previous state of the price itself without considering the microscopic factors behind it and for this purpose we do not specify a specific parametric model. In this research, our model will have the general form of a stochastic differential equation and the coefficients of this SDE will be extracted analytically based on Kramers-Moyal coefficients. Also, we show that the jump amplitude and jump rate can be extracted analytically based on higher moments. Our other contribution in this research is introduction of the quantity Q, which is calculated based on the 4th and 6th moments and is a practical criterion for choosing between the diffusion and the jump-diffusion models in the real world. We evaluate the accuracy of our method through the reconstruction of a known process and we also provide a practical application of KM method in finance.In one of the practical applications presented in this research, the importance of extracting process dynamics compared to extracting its probability distribution is evident. By extracting the probability distribution of a process and then sampling from it, we can answer only the questions where temporal order and sequencing is not important, for example calculating VaR over a given period. But the problem of first passing of the process value from a specific level in a given day is an example of a problem that shows the importance of extracting the dynamics of a stochastic process because two processes with the same probability distribution can have different dynamics. For example, the simple Ornstein-Uhlenbeck process with a drift coefficient of $-\boldsymbol{x}$ and a diffusion coefficient of 1 has a normal probability distribution ~N(0,1) and at the same time a complex stochastic process with drift coefficient $-\boldsymbol{x}^3+\boldsymbol{x}$ and diffusion coefficient $\boldsymbol{x}^2+1$ also has the same distribution [9]. That is, by having the probability distribution of the return, the probability of a specific level of loss can be calculated, but to calculate the probability of the occurrence time of such a loss, the dynamics of the process must be available. Considering that if there is a jump in the stochastic process, in the general case there is no analytical solution for the first passage time distribution, then for a practical case we have obtained this distribution by simulating the extracted dynamics. In problems related to stochastic processes, the passing of the process value from a specific level at a certain time is known as the first time passage problem. In this research, we will use this concept and the mathematical tools available in this field to calculate the probability distribution of the occurrence of a certain level of loss or return.Next, in the second part, we will have the theoretical foundations and literature review, and in the third part, the Kramers-Moyal coefficients and their relationship with the parameters of the Langevin equation will be presented. In the fourth part, we will have the practical extraction of Ethereum cryptocurrency dynamics, and finally, the fifth part is dedicated to the conclusion.

2 Foundations and Literature review

The observed jumps in the interval between two external shocks are additional fluctuations observed in asset prices that cannot be explained by the changes in its intrinsic value. Put it differentially, when an exogenous shock occurs, due to the change of fundamental factors and as a result of the change in the expected future cash flows, the intrinsic value of the assets also changes but the amount of price fluctuation observed in the market is not consistent with the estimated change in intrinsic value. This problem is known as the excess volatility puzzle in financial literatures [10]. Existing theoretical foundations to explain the phenomenon of excess volatility observed in asset prices can be divided into two main categories:

1) The first category is based on the formation of emotions and the occurrence of overreaction (or underreaction) in the behaviour of market agents at the time of shock. Among the famous theorists in the field of behavioural finance, Robert Shiller, a Nobel Prize winner in 2013, considers economic narratives formed in the social networks as the factor that stimulates market agents

[6]. In his opinion, in the light of the existence of a positive feedback mechanism in the behaviour of market agents, a small noise can be intensified and become an effective shock in forming a price bubble. The amplification of small shocks and their transformation into bubbles and large price fluctuations has been proposed by George Soros, a famous capital market activist, in another theory called the "reflexivity theory". The idea of reflection, which indicates the existence of a circular behaviour between agents in financial markets, was implemented in a model called LPPL(Log-Periodic Power-Law) which inspired by the behaviour of spins in statistical physics and in fact it is an application of the Ising model in the field of finance[11]. In two separate domestic studies [12,13], the LPPL model has been used to explain the excess fluctuations observed in prices on the Tehran Stock Exchange, and the results show that the aforementioned model is capable of modelling the positive feedback mechanism in the behaviour of market agents with good accuracy.

2) The second group are theories that still based on the maximization of marginal utility and try to explain the excess volatility by considering the household risk aversion rate as a time varying parameter under macroeconomic shocks [14] or by adding the concept of ambiguity aversion to the utility function of household [15]. The study of this type of models shows that the goal is still to derive the fundamental equation (According to Cochrane [16], the basic equation for pricing capital assets is $E(M_{t+1}R^e_{t+1}) = 1$ where $R^e_{t+1} = r^e_{t+1} + R_f$ is the total return. r^e_{t+1} is the risk premium and R_f is the risk-free return and $M_{t+1} = \beta\left(\frac{U'c_{t+1}}{U'c_t}\right)^{-\gamma}$) for capital asset pricing using the Euler equation, with the difference that in these types of models, the household utility function is considered with a time-varying risk aversion rate or with a new parameter named ambiguity aversion in order to achieve a model that is capable of generating the excess volatility observed in real data.

An examination of the models presented based on the above theories shows that the main goal of these models is to produce an unusual and large amplitude of fluctuations. While in addition to the problem of large amplitude of fluctuations in observations, there is also the problem of the existence of high-frequency jumps, so that this point has not been considered in previous models. However, in this research, as an innovation, a model is extracted that is able to generate the large frequency observed in the data in addition to generating a large amplitude of fluctuations.Studying of the previous researches shows that various models have been proposed to investigate the excess volatility puzzle, including behavioural models [7,17,18] the time-varying risk aversion model[14,16], intermediary asset pricing models[19,20], ambiguity aversion models [21,22], rare event models[23] and models based on growth probability assessment[24,25].Based on the data, the long-term mean of stock risk premium is about 7%, which fundamental factors cannot explain it[10]. in fact, the long-term mean of stock risk premium what is obtained based on fundamental factors, is about 4%. One of the models presented to solve the observed contradiction between the real data and the results of the models is the time-varying risk aversion model [14,16], which based on this model the difference between the two values of the observed risk premium and the calculated risk premium from the Euler equation-in the household consumption optimization problem- refers to the difference in the rate of risk aversion during boom and recession. This means that during the period of boom, the risk aversion of the household is lower than it during the period of recession, and this causes households to have a lower expected return from risky assets during the period of prosperity, and as a result, during this period the demand for risky assets and, of course, their price increases. this mechanism makes the stock risk premium more than what can be explained based on consumption growth. Another solution that has recently been proposed to solve this puzzle is the intermediary asset pricing model. In this type of models, it is claimed that the risk-bearing capacity of financial intermediaries depends on the possibility and opportunity of financial leverage by them, and therefore changes in the risk-bearing capacity of financial intermediaries in light of changes in financial leverage opportunities cause asset price fluctuations[19,20]. Also, models based on the concept of ambiguity aversion have been introduced to solve the puzzle of excess volatility, and based on this view, increasing uncertainty about the expected return of an asset causes its price to decrease. As variable risk aversion models, it is claimed that the ambiguity aversion can have a similar effect on the asset price. This means that if the uncertainty about the expected return of an asset increases, then the investor in addition to risk premium also demands ambiguity premium, and therefore, in order to compensate for this uncertainty, the price of the asset will decrease more than in the case where there is only the effect of risk aversion[15].Along with the models presented by economists to investigate excess volatility in financial markets, researchers from the field of physics and complex systems have also presented models to explain this phenomenon by adding the concept of endogenous shocks to the problem, which come next. After the introduction of the basic Black-Scholes model, various models have been introduced for pricing options using stochastic differential equations, including models such as Mikhailov-Nogel model [26] or Kurganov-Tadmor model [27] ,... . But the research by Halperin [28] is based on the simulation of the Langevin equation and therefore it is closer to our research. In the mentioned research, after applying assumptions on the form of the potential function and calibrating the coefficients of the Langevin equation, the resulting model as generalization of the Black-Scholes model has been used to predict the option price. In the research[29], which was performed by examining stock-specially news in a five-year period on 300 stocks, It is claimed that the properties of exogenous jumps are different from the properties of endogenous jumps that are observed in the absence of stock-specific news, so that exogenous jumps are sudden-which is acceptable considering the randomness of their source, i.e. news-but the characteristic of endogenous jumps is that they follow an accelerated growth in fluctuations and are also affected by the system's memory and are predictable. In two separate studies [11,30] about endogenous jumps, the occurrence of these types of jumps are explained based on the existence of a positive feedback loop and the amplification of noise in the behavior of market agents. The results of the mentioned researches show that, a small external shock is amplified in the long term with a factor of 5 in the stock market [11,30] and with a factor of 2 [11] in the foreign exchange market, due to the existence of positive feedback loop. Such a result has been investigated and confirmed with a different method from the mentioned researches by using instrumental variables and under the Inelastic Markets Hypothesis [31]. Based on this hypothesis, if an external shock causes to enter one percent of new capital to the stock market, then we will see a five percent increase in the stock market value in a one-year horizon. Now, based on what has been said, the diffusion-jump model is introduced. Relying on the results of the aforementioned researches and accepting the existence of weak and strong positive feedback mechanisms in different phases of the market, price fluctuations can be considered to include two parts of diffusion and jump, so that the diffusion part represents the fluctuations that are formed due to the mechanism of weak positive feedback, but the endogenous jumps are formed due to the presence of strong positive feedback in the behaviour of agents. A basic model for modeling jumps in financial markets is the Merton's jump-diffusion model[32], which is a generalization of the Black-Scholes equation and has the following form:

$$\frac{dS}{S} = (\alpha - \lambda k)dt + \sigma dZ + dq \tag{1}$$

where α is the expected return, σ^2 is the return variance, Z is the Wiener process, q is the Poisson process and λ is the average arrival of jumps per unit of time. Unlike the Merton's model, in which the form of the coefficients is considered as a default, in this research we want to extract these coefficients in a data-driven approach without any default form. Since the Merton's jump-diffusion model is a special case of the generalized Langevin equation, examining the aforementioned model in general and without any additional assumptions about the form of its coefficients will be equivalent to examining the generalized Langevin equation. Fortunately, the solution of basic and generalized Langevin equation have been investigated in detail and even in higher dimensions by mathematicians and physicists [33,34,35] and extracting the parameters of these equations has been investigated by various methods. The results of the research [36] show that among the Kramers-Moyal, kernel, maximum likelihood and histogram methods for extracting the coefficients of the generalized Langevin equation, the Kramers-Moyal method is superior to the other mentioned methods, and this superiority is especially evident when the sample size is small. Therefore, our main goal in this research is to apply the results of physics re-searches in finance to remove the gap between the tools for modelling stochastic processes in physics and finance. Other findings suggest that cost stickiness has a positive impact on the relationship between institutional investors and passive institutional investors with conservatism [52]. The findings of some researchers showed that there is a significant relation between the stock market uncertainty changes in an economic boom and the investment risk in general, which is not significant in terms of the economic turndown. The Investment risk during both economic boom and recession is decreased by the unexpected increase in profit of each share and propagation of positive news. Although the risk is increased by the spread of negative forecasts in relation to shares [53]. The researchers' findings show that risk premium was a determining factor in explaining changes in investors' expected rate of return, and that there was a conditional relationship between the Downside Beta and expected return. Therefore, to explain the relationship between risk and return, one must pay attention to the market direction [54].

3 Methodology

In this section, we first describe the method of estimating the diffusion model coefficients based on the Kramers-Moyal coefficients, and then we generalize the results to the jump-diffusion model and examine the effect of the presence of jump on the drift and diffusion coefficients, also we provide a practical tool to choose between the diffusion model and the jump-diffusion model. It is claimed that the estimation of the coefficients of the jump-diffusion model by the KM method has better results than other estimation methods such as kernel method [36]. In comparing our non-parametric method with the parametric methods such as Hidden Markov Models, branching models and Variational Inference models, we can say that in the mentioned parametric models, a default distribution with unknown parameters is considered for the hidden state of the system and also for the secondary process that rides on the hidden state. For example, in the Hidden Markov Models[37,38], a Bernoulli distribution with unknown parameter P is considered for the hidden states, and a normal distribution with unknown mean and variance considered for the secondary process and these parameters are usually estimated by maximum likelihood method. But estimating the parameters of Hidden Markov Models by assuming distributions other than the Bernoulli and Normal distributions, requires heavy computational work, and if in these models we want to consider the jumps observed in the data, the problem of parameter estimation will be more complicated. But in our non-parametric method, all estimations are only based on the calculation of conditional moments.

3.1 Estimation of coefficients of diffusion and jump-diffusion model

Our unknowns in diffusion and jump-diffusion models are the coefficients of SDEs, which are known as the Langevin equation and the generalized Langevin equation in statistical mechanics, respectively. We claim that the parameters of theses equations can be extracted based on the coefficients of Kramers-Moyal equation. Kramers-Moyal equation is a PDE that describes the probability distribution of Markov processes. This PDE is a differential equation of first order in time and infinite order in the state variable of the system. The steps of deriving this equation are described by details in references such as [33,34,35]. The general form of this equation is as follows:

$$\frac{\delta P(x,t)}{\delta t} = \sum_{n=1}^{\infty}\left(-\frac{\delta}{\delta x}\right)^{n} D^{n}(x,t)P(x,t) \quad (2)$$

The terms $D^n(x,t)$ in Eq.(2) are called Kramers-Moyal coefficients. According to the above equation in general, Kramers-Moyal coefficients are the coefficients of an infinite series, which shows the first order time evolution of probability of the system state($\frac{\partial P(x,t)}{\partial t}$) in terms of the spatial evolution of the higher order probability of the system state($\frac{\partial^n P(x,t)}{\partial^n x}$). According to relation 3 in below, Kramers-Moyal coefficients can be calculated in terms of conditional moments:

$$D^{n}(x,t) = \frac{1}{n!}\lim_{\tau\to 0}\frac{M^{(n)}(x,t,\tau)}{\tau} = \frac{1}{n!}\lim_{\tau\to 0}\frac{1}{\tau}\langle[x(t+\tau)-x(t)]^{n}\rangle_{|x(t)=x'} \quad (3)$$

3.1.1 Pawula Theorem

Due to the infinite series in Eq.(2), it is practically impossible to solve this PDE. Therefore, it seems that we should look for special cases of this equation that will lead us to a limited series and thus to a solvable equation. Pawula's theorem states that there are only three possible case for the Kramers-Moyal equation [39]:

1) n=1: the series of coefficients is truncated in the first term.
2) n=2: the series of coefficients is truncated in the second term.
3) n=∞: the series of coefficients does not stop.

The summary of Pawula's theorem is that if we accept the assumption $D^2 = 0$, then the series is truncated in the first term, and this means that the system is deterministic. If we accept $D^2 \neq 0$ and $D^4=0$, then the series will have a maximum of two terms and the process is stochastic, continuous and diffusive. If $D^4 \neq 0$, then the series will not be truncated, and therefore the process is discontinuous, and in addition to diffusion, it also has jumps.

3.1.2 Langevin equation and its relationship with Fokker-Planck equation

In the special case of $D^2 \neq 0$ and $D^4=0$, the KM equation is called the Fokker-Planck equation:

$$\frac{\delta P(x,t)}{\delta t} = -\frac{\delta}{\delta x}\left(D^{1}(x,t)P(x,t)\right) + \frac{\delta^{2}}{\delta^{2} x}\left(D^{2}(x,t)P(x,t)\right) \quad (4)$$

The general form of the Langevin equation is as follows:

$$\frac{dx(t)}{dt} = \alpha(x,t) + \beta(x,t)\eta(t) \quad (5)$$

Where $\alpha(x,t)$ is called the drift term and $\beta(x,t)$ is the diffusion coefficient, and $\eta(t)$ is a white noise with zero mean and Dirac delta correlation function. The Wiener process is a special case of the Langevin equation, which is obtained by choosing the values $\alpha(x,t)=0$, $\beta(x,t)=1$ and $x(t)=W(t)$:

$$\frac{dW(t)}{dt} = \eta(t) \tag{6}$$

According to Eq. 5 and 6, the Langevin equation can be written in the following form in terms of the Wiener process:

$$dx(t) = \alpha(x,t)dt + \beta(x,t)dW(t) \tag{7}$$

Now we express the relationship between the Langevin equation and the Fokker-Planck equation with this proposition: "If the dynamics of a stochastic process follows the Langevin equation, then the probability distribution function of such a process applies to the Fokker-Planck equation". The proof of this proposition and also the relationship between the coefficients of the Fokker-Planck equation and the coefficients of the Langevin equation, which is as follows, is given in [35]:

$$D^1(x,t) = \alpha(x,t) \tag{8}$$

$$D^2(x,t) = \frac{1}{2}\beta^2(x,t) \Rightarrow \beta(x,t) = \sqrt{2D^2(x,t)} \tag{9}$$

Relations (8) and (9) state that in order to extract the coefficients of the Langevin equation, the coefficients of the Fokker-Planck equation are needed which can also be calculated from the conditional moments.

3.1.4 jump-Diffusion model or generalized Langevin equation

The reason for examining the Langevin equation and it's generalization in this research is that most of the models in finance are a special form of the Langevin equation, including the Black-Scholes equation for analyzing stock price dynamics, the CIR (Cox-Ingersoll-Ross) equation for analyzing interest rate dynamics and Heston model to study the volatility in financial markets. Also, the jump-diffusion models that today are used for price modeling with various limits about its coefficients, in fact are the generalized Langevin equation, which includes the Poisson process in addition to the Wiener process. Based on the Lindberg continuity criterion [35], the Langevin equation itself is considered a continuous model, but based on the aforementioned criterion, the generalized Langevin equation is a discontinuous model, and therefore it is a suitable option for modeling jump-diffusion processes.Inspired by the Merton model that was mentioned in the previous section, the jump-diffusion model that we will examine in this research is in the form of generalized Langevin equation [40]:

$$dx(t) = \alpha(x,t)dt + \beta(x,t)dW(t) + \xi dN(t) \tag{10}$$

The term ξdJ(t) in the relation 10 is the difference between the Langevin equation and the generalized Langevin equation and is given to show the jumps. J(t) is a Poisson process with jump rate λ(x,t), $\xi \sim \mathbf{N}(0, \sigma_\xi^2)$ represents the size of the jumps and σ_ξ^2 is the jump amplitude. In terms of financial concepts, when both of the amplitude and rate of jumps are large, it is equivalent to the existence of fat tails in distribution of studying process (i.e high probability of large losses).The drift coefficient represents the mean of changes in return in the short term between two exogenous shocks. The diffusion coefficient indicates the intensity of normal fluctuations or fluctuations of return in a period of time evolution of the process when weak positive feedback dominates the behavior of market agents. In contrast to diffusion coefficient, the jump amplitude is related to abnormal changes of the process or changes in the time of the Poisson event, where the Poisson event in this problem is the phenomenon of the illiquidity (the inability to quickly buy and sell an asset without changing its price).The market impact is defined as the average change in the price of an asset from the start to the end of a large transaction. Large

Poisson event equals to more illiquidity and large market impact in low depth markets[41]. Therefore, according to the definition of the market impact, the jump amplitude in the generalized Langevin equation is a good option to quantify market impact or depth of market.
Now we want to see if there is a jump in the process, are the drift and diffusion coefficients the same as the previous values that were extracted in the basic Langevin equation, or does the presence of the jump cause these coefficients to change?

The unknowns in the jump-diffusion model are extracted based on the first, second, fourth and sixth moments as follows [35]:

$$\alpha(x,t) = \frac{M^1(x,t)}{\tau} = D^1(x,t) \tag{11}$$

$$\beta(x,t) = \sqrt{\frac{M^2(x,t)}{\tau} - \frac{\left(M^4(x,t)\right)^2}{3M^6(x,t)}} = \sqrt{2D^2(x,t) - \lambda\,(x,t)\sigma_{\xi}{}^2(x,t)} \tag{12}$$

$$\sigma_{\xi}^2 = \frac{M^{(6)}(x,t)}{5M^{(4)}(x,t)} \tag{13}$$

$$\lambda(x,t) = \frac{M^{(4)}(x,t)}{3\sigma_{\xi}^4} \tag{14}$$

According to relations (11) and (12), in the presence of jump in the process, the drift coefficient is still equal to the first Kramers-Moyal coefficient, but the diffusion coefficient is different from its value in the pure diffusion process.

3.1.5 A practical criterion for choosing between the diffusion and jump-diffusion model

To choose between the diffusion model and the jump- diffusion model, based on theory, we referred to the Pawula theorem and claimed that if the fourth KM coefficient becomes zero, then there will be no KM coefficients higher than the second order, and therefore the dynamics of the process fallows the diffusion equation and otherwise, the jump- diffusion model should be chosen to model the studied process.But from a practical point of view, Pawula's criterion may cause mistake in model selection, and the reason for this is due to the problem of discretization of the studied process and the limited rate of sampling in practice.

That is, even if the original process is theoretically continuous, but by sampling with a finite frequency, our observations will have jumps, and depending on the sampling rate, there may be large-amplitude jumps in the observations, while there are no such jumps in the real process. And therefore, we may wrongly identify a continuous stochastic process as discontinuous. As a result, the non-zero fourth KM coefficient is not a strong reason for the discontinuity of the underlying process.
As a practical solution to correctly identify the type of the underlying process, a relation in the following form between the second and fourth conditional moments is extracted for a jump-diffusion process [42]:

$$\frac{M_J^{(4)}(x,t)}{3\left(M_J^{(2)}(x,t)\right)^2} \simeq \frac{\langle \xi^4 \rangle \lambda(x)\tau}{3\left(\beta^2 + \langle \xi^2 \rangle \lambda(x)\right)^2 \tau^2} \sim \frac{1}{\tau} \tag{15}$$

According to the relation(15), for a jump-diffusion process, the ratio $\frac{M_J^{(4)}(x,t)}{3\left(M_J^{(2)}(x,t)\right)^2}$ diverges as τ decreases, while this ratio is equal to one for a pure diffusion process. Thus, if we calculate the mentioned

ratio from the data and draw it in terms of τ, and the resulting curve be a horizontal line, then it is claimed that the underlying process is a diffusion process, but if the resulting curve diverges by decreasing the value of τ, then it is claimed that the underlying process is a jump-diffusion process.
Another interesting criterion introduced in [42] to detect the type of process is the Q quantity. Using the relationships extracted for the fourth and sixth conditional moments the value of this quantity is as follows:

$$Q(x) = \frac{M^{(6)}(x)}{5M^{(4)}(x)} \approx \begin{cases} \tau\beta^2(x) & \text{for diffusive} \\ \sigma_\xi^2(x) & \text{for jump} \end{cases} \tag{16}$$

According to the relation(16), we expect that for a jump-diffusion process, the quantity Q(x) to be independent of τ and be a horizontal line, while for a pure diffusion process, Q(x) has a linear relationship with τ, and in fact, the slope of Q-τ curve is equal to the square of the diffusion coefficient in the pure diffusion process.

3.2 Evaluation of the accuracy of Kramers-Moyal method

In order to evaluate the accuracy of the presented method, we generate a synthetic signal whose dynamics follows the generalized Langevin equation introduced in the relation (10) as follows:

$$\begin{cases} dX_t = -10X_t + 2X_t dW_t + \xi dN \\ \xi \sim Normal(0,1), \lambda = 0.1 \end{cases} \tag{17}$$

The numerical solution method used to generate synthetic data from Eq.(17) is the Euler-Maruyama method [43] and the number of generated data is 10^7.Considering this synthetic signal as an unknown process, the first step in its reconstruction is the choice between the pure diffusion model and the jump-diffusion model as the underlying dynamics.
As we said, the behavior of Q quantity is a good criterion to choose between these two models. The Q diagram related to this process is shown in Fig. 1:

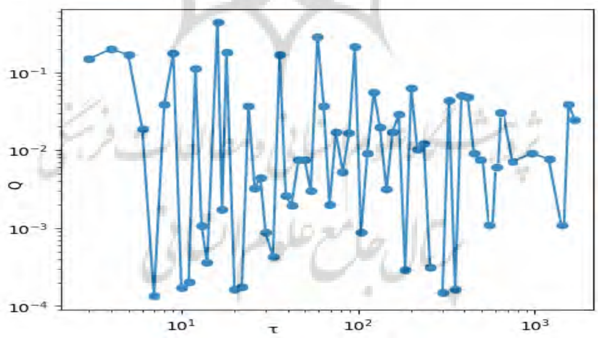

Fig. 1: Q-τ curve for synthetic data
Source: Research findings

According to the extracted Q diagram in Fig. 1, it can be concluded that the studied process should be modeled with the jump-diffusion model, and this result was expected considering the nature of our synthetic signal. The probability distribution used to calculate the moments has been extracted by the kernel density estimation method with Epanechnikov kernels[44,45,46]. Estimated results for jump rate and jump amplitude are given in Table 1:

Table 1: values for jump rate and amplitude

	λ	σ_ξ^2
Real value	0.1	1
Estimated value	0.1066	0.9480

Source: Research findings

It should be noted that all calculations and extraction of graphs in this paper were performed in the Python environment, and for example, the calculation of moments and density estimations required to extract the values of the jump rate and jump amplitude given in Table 1 were performed by statistical packages available in the scipy.stats library. The results of Table 1 show that the estimation of jump rate has a slight overestimation error compared to its true value, while the estimation of jump size has a small underestimation error. Performing several practical estimations with the conditional moments method shows that when using this method, the type of error for the jump rate and jump size parameters does not change, that is, in the case of jump size estimation, we always see an underestimation error and in the case of jump rate estimation, we always see an overestimation error. This point is useful in terms of practice in precisely adjusting the model parameters because, for example, after estimating the jump rate value from the data and considering the existence of an overestimation error for this parameter, it is possible to decide on the precise choice of the value of this parameter by adopting smaller values for it and performing a self-consistency test (which will be introduced in the next section).Graphs related to drift and diffusion coefficients based on the values expressed in Eq.17 (synthetic process), based on the values extracted by the KM method(moments ---), as well as the curves estimated for these coefficients from the KM outputs(regression) are given in Fig. 2 and 3:

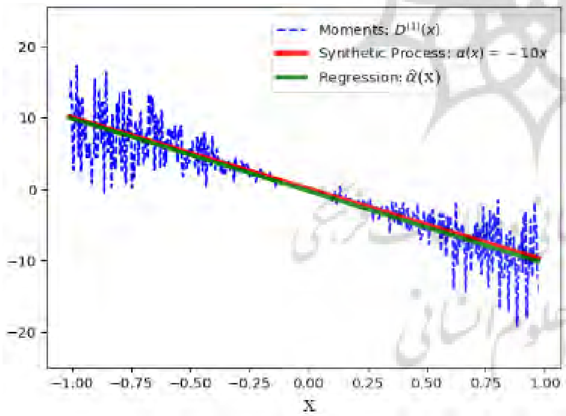

Fig. 2: Drift coefficient: estimated curve from KM method outputs(green) is good fitting for real value(red)

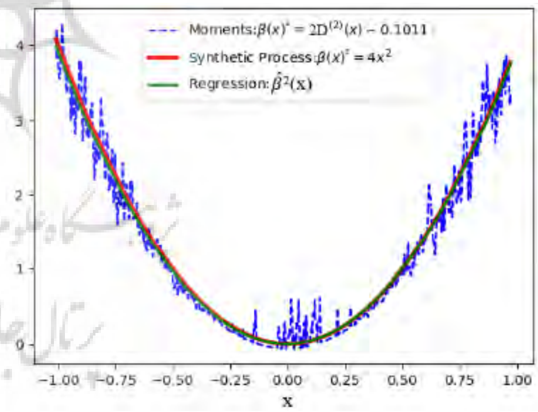

Fig. 3: Diffusion coefficient: extracted value $\hat{\beta}^2(x)=4x^2$ confirms the real value of $\beta(x)=2x$

Source: Research findings

According to Fig 2, the curve estimated based on the Kramers-Moyal coefficients(green) matches the actual curve $\alpha(x)= -10x$(red). Also, according to Fig 3, the curve $\hat{\beta}^2(x)=4x^2$ that estimated based on the Kramers-Moyal coefficients(green) confirms the actual value of the diffusion coefficient, namely $\beta(x)=2x$(red).

4 Practical applications of the KM method for extracting the ETH-USD dynamics

In this section, as an example of the application of the KM method in extracting the dynamics of real stochastic processes, we consider the price process of the Ethereum cryptocurrency. To extract the short-term price dynamics of this cryptocurrency, we get data from the cryptoarchive.com website for free by sampling rate one minute from October 17, 2017 to February 2, 2024.

The reason for choosing Ethereum cryptocurrency was that the Kramers-Moyal method requires a large amount of data to extract the parameters more accurately, and therefore, considering the age of this cryptocurrency compared to new cryptocurrencies and the existence of a sufficient number of observations for it (about 3 million data), the problem of limited data will be solved. Of course, the problem of having a sufficient data can also be solved by using Bitcoin data, and there is no preference between the two, but the reason for choosing this cryptocurrency instead of Bitcoin is the failure to obtain a stationary process even from the logarithmic return of Bitcoin. Since one of the conditions for using the Kramers-Moyal method is the stationarity of the process, it was preferred to use the price data of the Ethereum cryptocurrency because its logarithmic return is stationary.

Similar to the steps in Section 3.2, we again first determine the appropriate model for the underlying data generation process using the quantity Q. For this process, the behaviour of quantity Q with respect to τ is shown in Fig. 4:

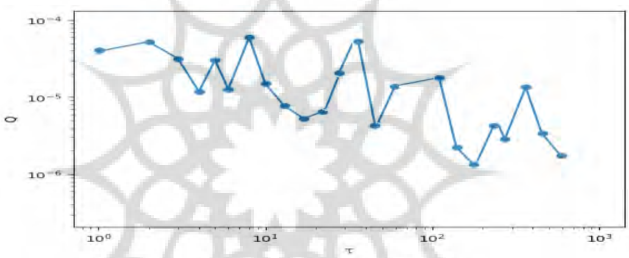

Fig. 4: Q-τ curve for real data
Source: Research findings

Again, according to Fig. 4, the quantity Q is horizontal and independent of τ, and therefore the appropriate model for this process is the jump-diffusion model. The extracted values for drift coefficient, diffusion coefficient, jump amplitude and jump rate for daily periods are listed in Table 2:

Table 2: parameters of jump-diffusion model

Drift coefficient	-1404x
Diffusion coefficient	31.3x
Jump Amplitude	0.00003
Jump rate	1290

Source: Research findings

The interpretation of the extracted drift coefficient is indirect and based on the definition of the potential function. The potential function U(x), which ①limits the behaviour of a process, ②forms stable and unstable equilibria, and ③determines the resilience of the system, can be extracted from the drift coefficient because we have [47]:

$$\alpha(x) = -\nabla U(x)$$

For the practical case studied and according to the results in Table 2, the drift coefficient has been extracted as a linear function, and therefore the corresponding potential function will be a quadratic parabola, and as a result, the process under study will have a stable equilibrium at zero point (Fig 5). Interestingly, the smaller the extracted drift coefficient and consequently the wider the resulting parabola for the potential function, we expect the amplitude of fluctuations around the equilibrium to be large, which is equivalent to a small decay rate, long mean revert time, and high entropy for the process. Conversely, the narrower the resulting parabola, we expect the amplitude of fluctuations around the equilibrium to be small.The extracted diffusion coefficient for the real process under study has a larger value than the diffusion coefficient of the synthetic process in the previous section, and therefore, by comparing this parameter for the two aforementioned processes, it can be concluded that in the real process under study, a stronger positive feedback loop is formed than in the synthetic process introduced. In other words, if the same noise is imposed to both systems above, then the noise amplification in the system with a larger diffusion coefficient will be more severe. In other words, when a shock occurs, the process with a larger diffusion coefficient will experience a larger amplitude of fluctuations around the equilibrium. According to the concepts expressed for the coefficients of the jump-diffusion model introduced in Eq. 10, the small jump amplitude extracted in Table 2 indicates that the market for this cryptocurrency is deep and has high liquidity, and in other words, the probability of the market being locked is low for this case. By implementing related SDE using the results of the Table 2 and solving it numerically, it is possible to generate the random trajectories of this process. The numerical solution method used to integrate the extracted dynamics, again is the Euler-Maruyama method as in the previous section. An example of the trajectory generated for the log-return along with the real data trajectory is shown in Fig. 5:

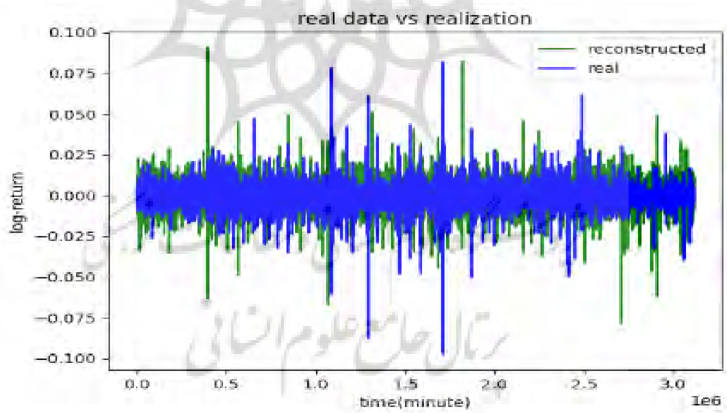

Fig. 5: sample reconstructed log-return vs real log-return

Source: Research findings

4.1 Self-consistency test to evaluate the accuracy of reconstruction

The self-consistency test is used to check the accuracy of the reconstruction of an unknown signal. In deterministic signals, after reconstruction we have a certain trajectory, but when reconstructing random signals, we will have different random trajectories of the reconstructed signal. To check the accuracy of a reconstruction, it is necessary to compare the probability distribution of the original signal with the reconstructed signal. Two well-known tests for self-consistency are the Kullback-Leibler divergence test[48] and the Jensen-Shannon divergence test[49,50]. In these tests, as much as the calculated divergence between the probability distribution of the original signal and of the reconstructed signal be closer to zero, it indicates the goodness of fitting of the coefficients so that the zero score for divergence indicates the two distributions are identical. KL-div is defined as follows:

$$KL(P||Q) = -\sum P_i \log\left(\frac{P_i}{Q_i}\right) = \sum P_i \log\left(\frac{Q_i}{P_i}\right) \quad (18)$$

The intuition for the KL divergence score is that when the probability for an event from P is large, but the probability for the same event in Q is small, there is a large divergence. When the probability from P is small and the probability from Q is large, there is also a large divergence, but not as large as the first case. That means in the KL-div test, it is different which distribution is considered as the reference distribution, and in other words, this test is not symmetric: KL(P||Q)≠KL(Q||P)

Unlike the KL-div test, the JS-div test although it is made from the KL-div but it is symmetrical. JS-div is as follows:

$$\begin{cases} JS(P||Q) = \frac{1}{2}\left(KL(P||M) + KL(Q||M) \right) \\ M = \frac{1}{2}(P + Q) \end{cases} \quad (19)$$

The advantage of the JS-div test compared to KL-div, in addition to its symmetry (JS(P||Q)=JS(Q||P)), is that the value of this quantity for the logarithm of base 2 is in the range [1,0]. JS=0 indicates the sameness of the two distributions and JS=1 indicates the dissimilarity of the two distributions.

Table 3 shows the results of the KL-div and JS-div tests to check the correctness of the reconstruction:

Table 3: Self-consistency test

KL-div	0.004
JS-div	0.036

Source: Research findings

The values close to zero for both self-consistency tests indicate the accuracy of the Kramers-Moyal method in reconstructing the unknown signal. In the following, an example of the random trajectories of the studied process (with specifications in Table 2) with the starting point x=2500 and in the time horizon of one day is showed in Fig. 6, and also by simulating the dynamics of this process 10,000 times, we get the probability distribution of daily losses as showed in Fig. 7:

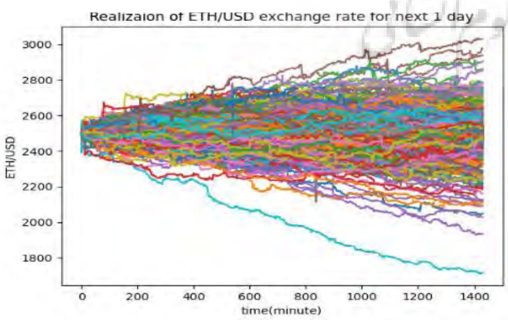

Fig. 6: Realizations of ETH/USD process

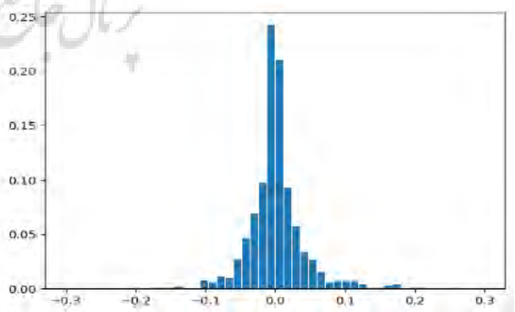

Fig. 7: Histogram of daily losses

Source: Research findings

According to the simulation results, the large range of daily fluctuations and the possibility of large daily losses (about 20%) indicate that this market is risky.The 99.9% quantiles for daily returns and daily losses, also the mean and standard deviation of daily returns were derived by the simulations performed 2164 times (equal to the number of days of real data) and the results along with the corresponding values for the real data are reported in Table 4:

Table 4: statistical results from real and simulated data

	Real value	simulation
99.9% quantiles for daily returns	0.2109	0.2286
99.9% quantiles for daily losses	0.2083	0.2089
Mean of daily losses	0.00079	0.00065
standard deviation of daily losses	0.0477	0.0436

Source: Research findings

The similarity of the simulation results with the actual values of the quantities stated in the table 4 is another confirmation of the accuracy of the performed reconstruction. Now, after ensuring the correctness of the reconstruction, common questions about stochastic processes can be answered using the extracted dynamics. The problem of passing the value of a stochastic process from a specified level in a given time is known as the first time passage (FTP) problem. The calculation of the probability distribution of the FTP for a pure diffusion process has been done analytically that, starting from the initial point x_0, the probability that x will cross the level L at time t is equal to [35]:

$$f(t) = \frac{|L - x_0|}{\sqrt{2\pi t^3}} e^{-\frac{(L-x_0)^2}{2t}} \tag{20}$$

But for the jump-diffusion processes, no such analytical solution has been provided in the general, so in this case, we turn to calculate the aforementioned distribution based on simulation. For the process under study, assuming that after an external shock the value of the process jumps to the level x=2500 and assuming that there is a sufficient time interval between two external shocks, then the probability distribution of first time passage from level x=3000, for a 30-day period and assuming that the market is in a rising phase, is obtained as shown in Fig. 8:

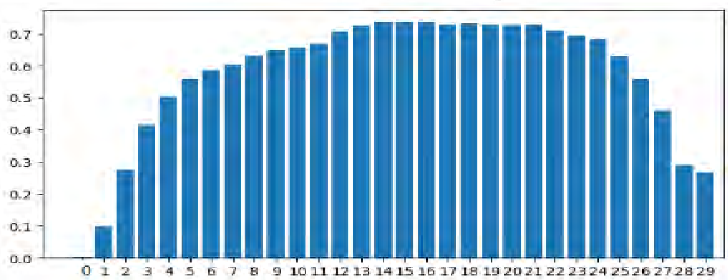

Fig. 8: probability distribution of first time passage from level x=3000
Source: Research findings

The results of the Fig. 8 that obtained by simulating the extracted dynamics 1000 times, show that with a 70% probability, the first passage of the asset price from the level x=3000 will be between the 14th and 21th days, and a look at the price chart of this asset on the Yahoo Finance website confirms this result, as according to the EHT-USD chart on Yahoo Finance, the price of this asset on February 10, 2024 was around $2500 and starting from this point, although the price touched the $3000 level on February 20, it was unable to break through this barrier, and the first certain passing of the asset price over the $3000 level will be on February 24, 2024. The recent application is an example of problems that show the importance of extracting the dynamics of a stochastic process compared to extracting its probability distribution, because as we have already stated in the introduction, two processes with the same probability distribution can have different dynamics and therefore, by only having the probability distribution of the process, many questions about its behaviour remain unanswered.
As another confirmation of the correctness of our reconstruction, we consider the EHT-USD chart in Fig. 9 from the historical data available on the Yahoo Finance website in the period of one month(red inset window) from 13April2024 to 13May2024. Considering that the mentioned period is between two big shocks, we expect our model to be able to explain the short-term dynamics observed in this period. Fig. 10 relates to the simulation results of the reconstructed process with the starting point x=3000 (the actual value of the process on 13April2024) and the time horizon of one month.

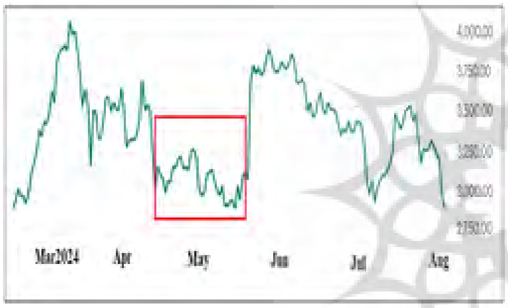

Fig. 9: ETH-USD historical chart

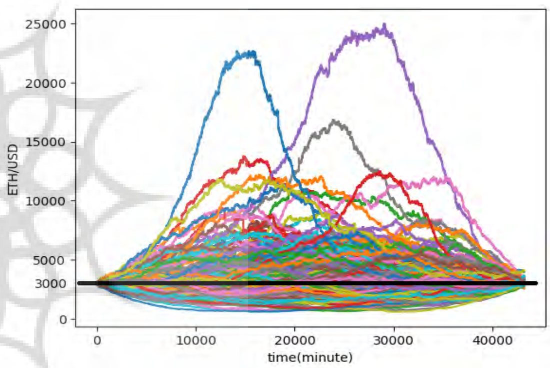

Fig. 10: simulation of extracted dynamics with an initial price of 3000$ and a time horizon of one month

Source: Research findings

The results of simulation show that the price of this asset can have different random paths during the period of this month but it will eventually revert to its initial value. To interpret the above result, we first recall that the Bloomberg website survey shows that no important news related to this cryptocurrency was published during the mentioned time period. Therefore, according to what we previously stated based on the standard economic perspective, after the first external shock and considering the rapid adjustment of prices based on this perspective, we expect a new equilibrium to be formed, which is the equilibrium value of $3,000. Again, as stated in the theoretical section, if price changes were only due to the external shocks, then in the interval between these two major shocks, we should have witnessed only a perturbative behavior in price movement, while the actual price chart shows a non-perturbative behavior with a large amplitude of daily fluctuations. This non-perturbative behavior (jumps) is endogenous and can occur even in the absence of major news related to the asset under study. Therefore, it is claimed that financial markets have non-equilibrium behavior and do not calm down. This means that even in the absence of economic news and only in the shadow of the formation of economic narratives, traders' fear & greed, and mass imitation of agents by each other, the market experiences irrational emotions. These emotions are amplified due to the existence of a positive feedback mechanism in the behavior of agents and create endogenous jumps, and it is said that "the market makes its own news". Farmer [8] considers the occurrence of the 2008 financial crisis to be an evident example of such a view, as incredibly and in a situation where no negative external shock (with a magnitude that would justify a market crash) had been reported, we witnessed a severe collapse in markets.The simulation results show that the extracted stochastic differential equation can well reproduce the jump-diffusion behaviour of the price process in the mentioned short-term period, and therefore the ability of the presented model in out-of-sample prediction is also acceptable because the data used to extract the process dynamics was up to February 2, 2024, while the prediction was made for the time after this date.

5 Discussion and Conclusions

In this research, instead of specifying a parametric model to generate the endogenous jumps, we considered general form of a SDE, the generalized Langevin equation, for the studied process, and in this sense, we claim that our method is a non-parametric method because we have not considered a special form for the drift, diffusion and jump components. Comparing our non-parametric method with parametric methods such as Hidden Markov Models shows that our method has more direct and lighter calculations than the mentioned parametric methods, because all our estimations are direct based only on the calculation of conditional moments. The use of the KM method to extract the parameters of the generalized Langevin equation in case of synthetic data, shows that this method is able to extract the unknowns with good accuracy. Researchers in the field of statistical physics that they are important users of the generalized Langevin equation show that the KM method works better than maximum likelihood, kernel and histogram methods in estimating the coefficients of the generalized Langevin equation and then considering the superiority of the KM method and since the jump-diffusion models used in finance are a special case of Langevin equation, therefore we decided to use this method for the first time in the financial field to extract coefficients of the jump-diffusion model. The result of using this data-driven method in the cryptocurrency world shows that, similar to the Merton model, here too, the drift and diffusion coefficients are a linear function of the system state variable, with the difference that in the Merton model these coefficients were assumed to be a linear function of the state variable by default, but here we have extracted them non-parametrically.From the practical point of view, after extracting the first KM coefficient, the potential function that limits behaviour of the process is also available. Also, using the results of this research and after the reconstruction and realization of the process in large numbers, the amount of value at risk can be calculated. Another important application of extracted dynamics is to calculate the probability distribution of the first time passage for jumpy processes. With the dynamics in hand, many other questions that we did not address in this research can be examined, including the average time of presence of the state variable around an equilibrium, calculating the optimal time to exit the market and...

Comparing our research with Halperin’s paper[28] show that in the mentioned research, the Langevin equation was extracted without considering the jump term and also the potential function is considered to be a polynomial of the fourth degree(and therefore the drift coefficient will be a polynomial of the third degree), but when we use the Kramers-Moyal method, we no longer need to apply constraints on the potential function and consider a specific form for it, because we are able to extract its exact form from the data. Although in Halperin's research also used the Langevin equation to model asset price dynamics, since the application of the extracted dynamics in his research is for option pricing, while we have examined the first time passage problem using the extracted dynamics, then the results are not comparable. As policy application of the extracted dynamics, we can refer to the problem of the time of the first passing of the process over a specific level. So, if losses of more than 20 percent in a financial market are considered as crisis, then calculating the probability distribution of the times when the amount of the loss exceeds the 20 percent level will be equivalent to calculating the probability of a crisis occurring on different days.While the previous application was related to finance, now we introduce an application of Kramers-Moyal method for policy making in economics. Given the possibility of extracting the potential function of a stochastic process in terms of the first Kramers-Moyal coefficient and the relationship between the resilience of a system and the form of the potential function, this point can be used in policy making. For example, in the discussions of the existence of multiple equilibria for the inflation, it is claimed that, contrary to the New Keynesian idea, inflation may have more than one stable equilibrium [51], and therefore it is necessary to be careful that when applying a short-term expansionary policy to create economic prosperity, if the amplitude of the inflationary shock exceeds a certain limit, it is possible that it will trap us in inflation around a higher equilibrium. Therefore, having the form of the potential function will be useful for policymakers to examine equilibrium points and determine the permissible range of the amplitude of inflationary shocks. After extracting the jump-diffusion model, it can be used to generate synthetic data in machine learning. In fact, since unlike the diffusion behavior, the presence of jumps in a stochastic process is a rare behavior, then it is not possible to collect historical data for jumping behavior in large numbers. The lack of real data for jump-diffusion processes is a fundamental problem for machine learning about these types of processes, because most machine learning methods require a large number of observations to better learn model parameters, and therefore generating synthetic data based on the jump-diffusion model can be a solution to this problem. In order to clarify the issue, suppose we are looking for an algorithm to predict the occurrence of market crashes based on historical data, but since market crashes are rare phenomena, therefore, using only historical data it is not possible to present an algorithm that accurately predict this phenomenon. But by simulating the extracted dynamics and producing synthetic observations that include the phenomenon of market crash, a better algorithm can be provided. In the end, as a roadmap for our future research in this field, we announce that in the next research, we first plan to use the more complex Levy process instead of the Poisson process in the generalization of the Langevin equation and compare the results of the current research with the result of the model based on the Levy process. And the continuation of our research will be in the direction of generalizing the one-dimensional Langevin equation to the two-dimensional case and using it in the financial field to study the Heston model, because the Heston model is a special case of the two-dimensional Langevin equation.

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