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New Criterion for Fractal Parameter in Financial Time Series

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ABSTRACT

Since calculating the amount of fractal in the ARFIMA time series and increasing its accuracy and bring it closer to reality is very important, this article intends to investigate the possibility of modifying this computational formula by changing the focus criterion and using simulation. In the present paper, by analysing and simulating the fractal parameter for time series ARFIMA model and redefining and reviewing the Fractal mathematical, a fractal calculus and dimension in comparison with Euclidean norms introduced. In this regard, first, a new criterion about fractal or Hausdorff component for measuring the forms of fractal time series introduced, then the effects and functional inquiries using simulation data searched, and some mathematical proofs through simulation of data achieved. The findings showed that, the deviation of the new estimator from the simulated initial value is less, and closer to reality as this new criterion introduced by changing the focus criterion and replacing the mean with the median due to less sensitivity to out-dated data. The new criterion is better for determining the fractal parameter and identifying its degree of effectiveness. Finally, the findings empirically indicated that the proposed criterion is more efficient and better than the others for calculating fractal dimensions.

1 Introduction

Many time series approaches in time domain, such as Box Jenkins patterns autoregressive integrated moving average (ARIMA patterns) [4] are essentially partial autocorrelation functions used to examine gradual evolution of time series with regard to parametric patterns [3]. In datasets about finances, past observations are associated with observations in distant future and their relation is permanent. Hence, time series contain long-term memory features and understanding these features has significant implications for both short-term sales and long-term investment strategies [7]. Investment strategies would depend on the amount of long-term memory [14]; since the decisions related to investment are reactive to time domain under study. Long-term memory causes a nonlinear reliance on the first torque of the efficiency distribution, producing a predictable parameter in time series dynamics. This parameter is termed long-term memory parameter [16] in financial literature. Long-term memory models, called

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ARIMA time series long-term memory models, represented by ARFIMA (Fractional ARIMA) symbol [18]. Over the past decades, long-term memory processes have been an essential part of analyses of time series [21]. Long-term memory mechanisms are distinguished by very slow autocorrelations or a spectral density with a peak at a near-zero frequency [22]. Such features dramatically alter prognostic predictive actions [35]. As a result, many of the theoretical results and methods used in short-term memory series analysis such as ARIMA processes are not suitable for long-term memory models [33]. The presence of long-term memory is both logically and empirically relevant in financial assets [44]. If market has long-term memory, a significant autocorrelation will occur between observations studied over a very long period. Due to the separation of time, knowing distant past tends to predict future, and the probability of continued anomalous increases [46].

The presence of long-term memory on financial market infringes poor aspect of market efficiency theorem, as well as contradicting linear asset pricing models raising this question whether non-linear models can be used in asset valuation [48]. New developments in investment strategies and improved transparency on the economy have put us back to efficient markets than ever before. Therefore, if time series have a long-range dependency function, their changes will be predictable [47]. ARFIMA (p, d, q) [8], autoregressive fractionally integrated moving average process has been widely used in various fields such as astronomy, hydrology, mathematics, and computer science to represent a time series with long memory property [9]. A wide range of estimators for fractional parameter d recently appeared in time series literature. Fractal analyses of time series, transects, and natural or man-made structures have seen broad uses in almost all scientific disciplines [30] Actually, we are thinking about fractal or Hausdorff dimension quantifying roughness or smoothness of time series and spatial details in boundary as the perceptual scale is indefinitely perfect. In reality, measurements can be produced on a limited range of scales; thus, practical calculations of fractal dimension rely on the availability of observations with a reasonably fine temporal or spatial resolution. In general, the d-estimators can categorize into two parametric and semi-parametric group methods.

Methods involving probability function are the most common ones in first group. All parameters (self-regressive, moving average and differentiation) can be estimated at same time in parametric approach. In semi-parametric methods, the parameters estimated at two stages: only d estimated in first step and the others estimated in second step. Because Gaussian parametric estimations for long memory span based time series models have been rigorously validated providing an attractive alternative to semi-parametric approaches. Nonetheless, Gaussian parametric methods require a large amount of computing whereas semi-parametric techniques are easy to implement and less computationally demanding [24]. ARMA systems are short-memory mechanisms in sense that an impact at a given time is not permanent and they do not affect possible progression of time series. Infinite memory processes such as Difference Stationary (DS) have an opposite nature. This dichotomy is insufficient to account for long-term phenomena as demonstrated by the work of Hurst (1951) [19] in the field of hydrology. A long, but finite, memory process is an intermediate case, in that the impact of a shock has long-lasting consequences for the future values of time series; however, it will find its natural level of equilibrium. [27] formalized this type of behavior based on fractional Brownian movements and fractional Gaussian sounds. From these studies, the fractional ARIMA process defined as ARFIMA. More recently, these processes extended to seasonal cases and referred to SARFIMA process [49]. The main idea in this paper is to introduce a new criterion for calculating the time series types, investigate results by using simulation, and compute all parameters of ARFIMA process based on the iterative methods. For this analysis, we consider several definitions of fractional calculus and Hausdorff measure in Section 2. In section 3 we

describes the Random or chaotic tests for time series and fractal dimension and Euclidean norm again to prepare ARFIMA model are explained in Section 4. All results of a simulation study on ARFIMA models in R present in Section 5 and Section 6 gives a summary and some concluding remarks.

2 Fractional Calculus Definitions

This section includes fractional derivatives and definitions [37]. First, we define a fractional calculus to classify the fractional dimension classically [38]. Essentially, three useful and widely used fractional operators presented here, and we recall them as Caputo, Caputo – Fabrizio, and the Atangana – Baleanu derivative [1]. Let a function $w : R^+ \rightarrow R$ with fractional order$\alpha > 0$, is not integer, defining the fractional integral of the order$\alpha > 0$ is as follows:

$$I_t^{\alpha}(W(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\psi)^{\alpha-1} W(\psi)\, d\psi \tag{1}$$

Here Γ describes the Gamma function and α shows the fractional order parameter. If α is integer value then Function

$$\Gamma(\alpha) = (\alpha-1)! = \begin{cases} \int_0^{\infty} t^{\alpha-1} e^{-1} dt & if \alpha > 0 \\ \infty & if\ \alpha = 0 \\ \frac{\Gamma(1+\alpha)}{\alpha} & if\ \alpha < 0 \end{cases} \quad \text{and } \Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}} \tag{2}$$

The Caputo derivative for the given function $w \epsilon C^n$ with order α described is as follows:

$$C_{D_t^{\alpha}(w(t))} = I^{n-\alpha} D^n w(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{w^n(\psi)}{(t-\psi)^{\alpha-n+1}}\, d\psi \tag{3}$$

and$n-1 < \alpha < n \epsilon N$. Clearly, $C_{D_t^{\alpha}(w(t))}$ tends to $w'(t)$ as $\alpha \rightarrow 1$

Suppose $w \epsilon H^1(p,q)$, with $q > p$, and$0 \leq \alpha \leq 1$, then the definition of Caputo–Fabrizio derivative is follows as:

$$D_t^{\alpha}(w(t)) = \frac{p(\alpha)}{1-\alpha} \int_a^t w'(\psi) exp[-\alpha \frac{t-\psi}{1-\alpha}] d\psi \tag{4}$$

Where $p(\alpha)$ denote the normalized function and holds $p(0) = p(1) = 1$. If $w \notin H^1(p,q)$ then, the following is suggested:

$$D_t^{\alpha}(w(t)) = \frac{\alpha p(\alpha)}{1-\alpha} \int_a^t (w(t) - w(\psi)) exp[-\alpha \frac{t-\psi}{1-\alpha}] d\psi \tag{5}$$

Let $v = \frac{1-\alpha}{\alpha} \epsilon [0,\infty)$, $\alpha = \frac{1}{1+v} \epsilon [0,1]$,then equation given can be expressed is as follows,

$$D_t^{v}(w(t)) = \frac{p(v)}{v} \int_a^t w'(\psi) exp[-\frac{t-\psi}{v}] d\psi \quad p(0) = p(\infty) = 1 \tag{6}$$

Further,

$$\lim_{v \rightarrow 0} \frac{1}{v} \exp\left[-\frac{t-\psi}{v}\right] = \varphi(x\psi - t) \tag{7}$$

Consider $\alpha \epsilon (0,1)$, for a function $w(\psi)$ then we can write the integral of fractional order α is as follows,

$$I_t^{\alpha}(w(t)) = \frac{2(1-\alpha)}{(2-\alpha)p(\alpha)} g(t) + \frac{2\alpha}{(2-\alpha)p(\alpha)} \int_0^t w(\psi)\, d\psi \quad t \geq 0, \alpha \epsilon (0,1) \tag{8}$$

The remainder of the non-integer-type Caputo type order integral of the function with order $\alpha \epsilon (0,1)$ is

a mean into w with integral of order 1. Thus, it requires,

$$\frac{2}{2p(\alpha)-\alpha p(\alpha)}=1 \tag{9}$$

Implies that $p(\alpha)=\frac{2}{2-\alpha}$, $\alpha\epsilon(0,1)$. Then new Caputo derivative is suggested with $\alpha\epsilon(0,1)$ and is given by

$$D_t^{\alpha}\big(w(t)\big)=\frac{1}{1-\alpha}\int_0^t w'(x)\exp[-\alpha\frac{t-\psi}{1-\alpha}]d\psi \tag{10}$$

Consider $w\epsilon H^1(p,q)$, where q greater than p, and $0\le\alpha\le 1$, then we define the Atangana–Baleanu derivative in the following:

$${}^{ABC}_{a}D_t^{\alpha}w(t)=\frac{p(\alpha)}{1-\alpha}\int_a^t w'(\psi)E_{\alpha}[-\alpha\frac{(t-\psi)^{\alpha}}{1-\alpha}]d\psi \tag{11}$$

The fractional integral for the Atangana–Baleanu derivative expressed as follows:

$${}^{ABC}_{a}I_t^{\alpha}w(t)=\frac{(1-\alpha)}{p(\alpha)}w(t)+\frac{\alpha}{p(\alpha)\Gamma(\alpha)}\int_a^t f(\psi)(t-\psi)^{\alpha-1}\,d\psi \tag{12}$$

One can restore the original function for the case when $\alpha=0$. Some results regarding the Atangana–Baleanu derivative are pre- sented in the following:

The following is hold for a functionfϵC[a, b]:

$$\left\|{}^{ABC}_{a}D_t^{\alpha}\big(w(t)\big)\right\|<\frac{p(\alpha)}{1-\alpha}\|w(t)\| \tag{13}$$

where

$$\|w(t)\|=\max_{p\le t\le q}|w(t)| \tag{14}$$

The Lipschitz condition is satisfied by the Atangana–Beleanu derivative,

$$\left\|{}^{ABC}_{a}D_t^{\alpha}w_1(t)-{}^{ABC}_{a}D_t^{\alpha}w_2(t)\right\|<\varpi_1\|w_1(t)-w_2(t)\| \tag{15}$$

A fractional differential is given by the following equation,

$${}^{ABC}_{a}D_t^{\alpha}w(t)=W(t) \tag{16}$$

Possess a unique solution given by

$$w(t)=\frac{(1-\alpha)}{p(\alpha)}W(t)+\frac{\alpha}{p(\alpha)\Gamma(\alpha)}\int_a^t F(\psi)(t-\psi)^{\alpha-1}\,d\psi. \tag{17}$$

Let $N(\varepsilon)$ be the smallest number of closed balls of radius $\varepsilon>0$ needed to cover the set F. The fractal dimension of the set F is the number. Suppose for $\varepsilon>0$, an ε-cover of X is a finite or countable collection $\{B_i: i=1,2,\dots\}$ of balls $B_i\subset\mathbb{R}^d$of diameter $|B_i|$ less than or equal to ε that covers X. With

$$H^{\delta}(X)=\lim_{\epsilon\to 0}\inf\left\{\sum_{i=1}^{\infty}|B_i|^{\delta}:\{B_i: i=1,2,\dots\}\text{ is an }\varepsilon-\text{cover of X}\right\} \tag{18}$$

denoting the δ-dimensional Hausdorff measure of X, there exists a unique nonnegative value D such that Hδ (X) = ∞ if δ < D and Hδ (X) = 0 if δ > D. This value D is the Hausdorff dimension of the point set X. The Hausdorff dimension coincides with the box-count dimension, under weak regularity conditions,

$$D_{BC}=\lim_{\varepsilon\to 0}\frac{\ln(N(\varepsilon))}{\ln(\frac{1}{\varepsilon})} \tag{19}$$

Where N(ε) denotes the smallest number of.cubes of width ε in Rd which can cover X, and also with other natural and/or time-honored notions of dimension. The Hausdorff dimension, d, of a self-similar

set its connection to fractal geometry, though, as previously stated, there are many other applications of Hausdorff dimension which is scaled down by ratios $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$ satisfies the following two equations:

$$\varepsilon_1^d + \varepsilon_2^d + \cdots + \varepsilon_N^d = 1 \; and \; N\varepsilon^d = 1 \tag{20}$$

However, these equations do not appear in Hausdorff's paper, as they directly relate to fractals (and calculate the dimension of a fractal), which were ideas that Hausdorff would not have known about. Still, it's easy to see from these two equations how you can get a dimension that isn't a whole number, as

$$D(F) = \lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{\ln(\frac{1}{\varepsilon})} \tag{21}$$

3 Random or Chaotic Tests for Time Series

The most important measures that can be used in a time series framework to detect chaos are correlation dimension test [15] and Lyapunov exponent test [6]. Alternatively, the Dickey-Fuller test [10] used to verify the stationary characteristic. Another useful complementary test in this regard is the Hurst test [34]. One of the principal characteristics of chaotic processes is their high sensitivity to the initial conditions. This feature can therefore be used to detect chaotic processes [40]. Lyapunov exponent [26] is the best tool for detecting vulnerability to early conditions in a dynamic system. Positive Lyapunov exponent indicates exponential time path divergence, high sensitivity to initial conditions, and therefore, chaos. Negative Lyapunov exponent indicates exponential convergence of time paths and zero Lyapunov exponents indicates paths not diverging or converging.

In 1983, Grossberger and Prokasia presented later test of correlation. Examination of correlation dimension is a criterion for evaluating chaos theory in a time series process, and is a criterion for assessing a process's complexity. The higher the number measured in calculation of a time series correlation factor is, more complicated time series will be and more difficult it will be to be predicted. It must be ensured that it is stationary until modeling a collection of time. The Dickey Fuller test is among most useful stationary tests. It is typically inaccurate in sequence, because there is no set return standard [13].

3.1 Hurst Dimension

The Hurst method is based on studies carried out by Hurst to define the water mechanism at the Nile River dam. It was usually assumed that the water inflow into the lakes was natural, but Hurst after examining evidence from previous times inferred that there were recurring patterns to inflow. Moreover, method of study and Hurst test extended gradually to other phenomena appearing randomly but may have a regular pattern. In practice, Hurst exponent (H) can be estimated by performing a regression. The Hurst view is a useful tool to detect a non-random time series regardless of the type of distribution it contains. Hurst exponent, H, is defined as a function of the time span of a time series in terms of asymptotic behavior of the rescaled range. {\displaystyle R(n)}For self-similar time series, H is directly related to fractal dimension, D, where 1 < D < 2, such that D = 2 – H. Values of the Hurst exponent vary between 0 and 1, with higher values indicating a smoother trend, less volatility, and less roughness. Hurst exponent and fractal dimension can be separately defined for more general time series or multi-

dimensional systems, since Hurst exponent represents structure over asymptotically longer periods, while the fractal dimension represents structure over asymptotically shorter periods [44].

3.2 Estimating the Exponent

In literature, a number of long-range dependence estimators [11] were proposed. The oldest and best- known is the so-called rescaled range ($\frac{R}{S}$) analysis Popularized by Mandelbrot and Wallis and based on earlier Hurst hydrological findings. Alternatives include DFA, regression of the periodogram, aggregated variances, and local Whittle's estimator, wavelet analysis, both in the time domain and in the frequency domain [41].

3.3 Rescaled Range ($\frac{R}{S}$) Analysis

To estimate the Hurst exponent, dependency of the rescaled range on the time span n of measurement must be calculated first. A time series of full length N is divided into a number of shorter time series of length $n = N, N/2, N/4$, for each value of n, then the average rescaled range is calculated. Suppose Xt is a time series that produces a random stationary process [42] with $t = 1, \dots, n$.The statistics $\frac{R_n}{S_n}$ noted Qn is the extent Rn of partial sums of standard deviations of the series from its mean divided by its standard deviation Sn:

$$Q_n = \frac{R_n}{S_n} = \frac{\max_{i \le k \le n} \sum_{j=1}^{k}(x_j - \bar{x}_n) - \min_{i \le k \le n} \sum_{j=1}^{k}(x_j - \bar{x}_n)}{\left[\frac{1}{n}\sum_{j=1}^{n}(x_j - \bar{x}_n)^2\right]^{\frac{1}{2}}} \quad (22)$$

For a time series of length $n, X = X1, X2, \dots, Xn$ to calculate rescaled range the ($\frac{R_n}{S_n}$) algorithm we have to do the following steps:
Calculate the mean:

$$\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Create a mean-adjusted series:

$$Y_t = X_t - \bar{x}_n \ \ for\ t = 1,2, \dots, n$$

Calculate the cumulative deviate series Zi

$$Z_t = \sum_{i=1}^{t} Y_i \quad for\ t = 1,2, \dots, n$$

Compute the range R

$$R(n) = \max(Z_1, Z_2, \dots, Z_n) - \min(Z_1, Z_2, \dots, Z_n)$$

Compute the standard deviation S

$$S(n) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - m)^2}$$

Calculate the rescaled range ($\frac{R_n}{S_n}$) and average over all the partial time series of length n. The rescaled range (R/S) method is one of the time-domain analyses of Hurst parameter defined as follows:

$$\lim_{n\to\infty} E\left[\frac{R(n)}{S(n)}\right] = Cn^H \quad Be\ it \quad log(Rn/Sn) = log\ c + H\ log\ n, \tag{23}$$

Where E(0) denotes the expected value of the observations, the range of the first n values is R(n), the standard deviation is S(n), and the constant is C. Whittle's Maximum Likelihood Estimator (MLE) and wavelet analysis use Frequency Domain dependent period gram analysis [23]. In practice Hurst exponent (H) can be estimated by performing a regression. If value of Hurst exponent is equal to 0.5, this means an independent process of time series. If Hurst view is set between 0.5 and one, that implies a very long memory sustained time series. Eventually, if exponent of Hurst is equal to a positive value but less than 0.5, it means process is inconsistent. The H-value interpreted as follows:

If $0 < H < ½$ □ anti-persistent process.

If $H = ½$ □ a simple random process or ARMA process. There is a long-term dependence absence.

If $½ < H < 1$ long-term process, the dependence is even stronger as H tends towards 1.

The Revised $\frac{R}{S}$ Statistics have to revise to differentiate between long-range and short-range dependence, so that its statistical behavior is invariant over a general class of short memory processes but deviates for long memory processes. The following figures achieve this

$$\widetilde{Q_n} = \frac{R_n}{\hat{\sigma}_n(q)} \tag{24}$$

$$\hat{\sigma}_n(q) = \left[\frac{1}{n}\sum_{j=1}^{n}(x_j - \bar{x}_n)^2 + \frac{2}{n}\sum_{j=1}^{q}\omega_j(q)\left[\sum_{i=j+1}^{n}\left[x_j - \bar{x}_n\right]\left[x_{i-j} - \bar{x}_n\right]\right]\right]^{\frac{1}{2}} \tag{25}$$

$$\widetilde{Q_n} = \frac{\max_{i\le k\le n}\sum_{j=1}^{k}(x_j - \bar{x}_n) - \min_{i\le k\le n}\sum_{j=1}^{k}(x_j - \bar{x}_n)}{\left[\frac{1}{n}\sum_{j=1}^{n}(x_j - \bar{x}_n)^2 + \frac{2}{n}\sum_{j=1}^{q}\omega_j(q)\left[\sum_{i=j+1}^{n}\left[x_j - \bar{x}_n\right]\left[x_{i-j} - \bar{x}_n\right]\right]\right]^{\frac{1}{2}}} \tag{26}$$

This figure is different from the previous one by its denominatorQ_n, which takes into account not only the variances of individual terms but also the auto-covariance weighted associated with deviations of q as referred to: The calculation of H done as above and Lo analyses the behaviour of $\widetilde{Q_n}$ in the long-term dependence alternative. Then he shows up

$$V = \frac{1}{\sqrt{n}}\widetilde{Q_n} \xrightarrow{p} \begin{cases} \infty & pour\ H \in [0.5,1] \\ 0 & pour\ \ H \in [0,0.5] \end{cases} \tag{27}$$

Under hypothesis ofH_0: $X_t \overset{i.i.d}{\Longrightarrow} N(0, \sigma_x^2)$, there is a short memory in the time series ($H \in [0,5; 1]$). For the acceptance threshold at 5%, H_0 is accepted if $v \in [0,809\ ;\ 1,862]$. He concluded that: "For the values of H between 0,5 and 1 the acceptance Threshold of long memory at 10% is $v > 1,620$. For the values of H between 0,5 and 1 acceptance threshold of the anti-persistent hypothesis at 10% is $v > 0,861$." We can verify that there is a relation between the values d and the ARFIMA processes and H of the exponent Hurst ($d = H - 0,5$).

3.4 Redefinition Range ($\frac{R}{S}$) Analysis by Median

One of the advantages of median against mean is insensitivity to the extreme values of outliers. Whereas, like few outlier observations (too small or too large values, but not both), median provides

better than average value for the data measures of central tendency. One of the most important properties of the median is that the sum of the absolute values of the differences of the random variables is the minimum.

$$\sum_{i=1}^{N} |X_i - \text{median}| = \text{minimum} \tag{28}$$

Another important property of median data is that it is always between data mean and data mode.

$$mean \leq median \leq mode$$

or

$$mode \leq median \leq mean$$

Median calculates by order (calculation simplicity) and uses fast sorting algorithms to calculate very large or small amounts unaffected. On the other hand, disadvantage of influence of very large or small amounts and tendency towards them is that ability to calculate quantitative data is only the time-consuming calculation for long lists of numbers. Which one should be chosen based on introduction of three values for data centre trend measurement (arithmetic mean, median, and mode)? Many considerations are important when selecting right quantity for data centre calculation. Typically, our first option is the average, but there are cases in which median is better.

Median is rarely best value for data centre to represent. Median in statistics and probability theory is an indicator of the inclination towards the centre. The mean is a number dividing the statistical population or the probability distribution into two equal parts. One of significant benefits of medium to average is that median in the range of sizes not influenced by very large and very small numbers. Mean value is used in estimation of variance and standard deviation, and since data gap from median exceeds two, outliers have a greater impact on it. Median Absolute Deviation (MAD) specified to solve this problem. Calculate absolute magnitude of difference between all data, mean, and organize it from small to high. Sum of these variations is equal to the absolute value of the variance.

$$MAD = median(|X_t - median(X)|) \tag{29}$$

$$\overline{\overline{Q_n}} = \frac{\boldsymbol{R_n}}{\boldsymbol{S_n}} = \frac{\max\limits_{i \leq k \leq n} \sum_{j=1}^{k}(x_j - median(X)) - \sum_{j=1}^{k}(|x_j - median(X)|)}{MAD = median(|X_t - median(X)|)} \tag{30}$$

3.5 Fractal Dimension Again Euclidean Norm

Fractal analyses of time series, transects and natural or man- made surfaces have found broad applications in almost all scientific disciplines. While much of literature connects fractal properties to statistical self-similarity, there is no need for such a correlation. Rather, statement of fractal or Hausdorff dimension quantifies roughness or smoothness of time series and spatial data in boundary as observational scale is infinitesimally small [43]. In practice, measurements can only made at a finite range of scales, and practical estimates of fractal dimension rely on availability of observations with a sufficiently fine temporal or spatial resolution. We follow common practice in defining the fractal dimension of a point set $X \subset \mathbb{R}^a$ to be classical Hausdorff dimension. Fractal dimension then refers to properties of curve ($d = 1$) or surface ($d \geq 2$). It happens in continuum limit as the data are found in an infinitesimally dense

subset of temporal or spatial domain, which, without loss of generality, can be considered unit interval or unit cube. This restricting situation referred to as infill in time series analysis and spatial statistics. If boundary curve or boundary surface is smooth and distinguishable, its fractal dimension, D, is equal to its topological dimension, d. In case of a rough and non-differentiable curve or surface, fractal dimension can exceed the topological dimension. For example, suppose that $\{X_t: t \epsilon \mathbb{R}^d\}$ is a Gaussian cycle with stationary intervals, variogram or structure of which functions;

$$\gamma_2(t) = \frac{1}{2} E(X_u - X_{u+t})^2 \tag{31}$$

$$\gamma_2(t) = |c_2 t|^{\alpha} + O(|t|^{\alpha+\beta}) \text{ as } t \to 0 \tag{32}$$

where $\alpha \epsilon (0,2]$ $\beta \geq 0$ and $c_2 > 0$, and $|\cdot|$ denotes Euclidean norm. Then the graph of a sample path has fractal dimension

$$D = d + 1 - \frac{\alpha}{2} \tag{33}$$

This relationship relates fractal dimension of sample paths to action of variogram or structure function at origin of coordinates, and can generalize to a wide range of potentially anisotropic and non-stationary processes, as well as some non-Gaussian processes [36]. It allows us to consider the fractal dimension as a second-order property of Gaussian stochastic cycle, in addition to being a measure of roughness for a realized curve or surface [28]. Consequently, index in asymptotic relationship referred to as fractal index. Fractional Brownian motion is a non-stationary, statistically self-similar process defined in terms of variogram. Other table entries apply to stationary processes with covariance function. $\sigma(t) = \text{cov}(Xu, Xt + u)$, Which relates to the variogram as?

$$\gamma_2(t) = \sigma(0) - \sigma(t), \quad t \epsilon \mathbb{R}^d \tag{34}$$

4 ARFIMA Model

ARMA processes are short-memory processes in sense that shock at a given time is not sustainable and does not affect the future evolution of time series. Infinite memory systems, such as stationary variance processes, have an opposite behaviour: shock effect is permanent and influences possible values of time series [29]. To calculate degree of coupling, formula for rescaled range (R / S) analysis given to estimate level of coupling, which is now, called parameter Hurst. In addition, many valuable estimators of Hurst parameters provided to characterize LRD time series more precisely. Since then, numerous research studies have attracted the LRD, or long memory phenomenon.

Based on Hurst's study, more suitable models, such as ARFIMA and fractional integral generalized autoregressive conditional heteroscedasticity (FIGARCH) were built to evaluate LRD processes accurately. Analysis of autocorrelation function (ACF) is one of the useful techniques for recognizing patterns and periodicities in the data, in a way that is often more precise than simple visual inspections can be obtained. Additionally, ACF may define the LRD or long memory property. Let $\{X(t); t \epsilon(-\infty, +\infty)\}$ and ACF $\rho(k)$ is defined as:

$$\rho(k) = \frac{\mathrm{Cov}(X_t, X_{t-k})}{\mathrm{Var}(X_t)} \tag{35}$$

Where $\mathrm{Cov}(0)$ is covariance and $\mathrm{Var}(0)$ is the variance. A stationary time series defined over $t = 0,1,2,3$ ...is said to be long memory if $\sum_{k=0}^{\infty} |\rho(k)|$ diverges, where $\rho(k)$ is ACF of the process. Otherwise, time series said to be short memory or SRD. Another definition of long memory if for some frequency, $f\epsilon\,[0\,, 0.5]$, power spectrum $P(f)$, becomes unbounded.

$$P(f) = \int_{-\infty}^{\infty} e^{-2\pi i f k}\,\rho(k)dk \tag{36}$$

Where $-\infty < f < \infty\,, i = \sqrt{-1}$ and $\rho(k)$ is the ACF. The spectral density $S(f)$ is a normalized form of$P(f)$, defined by:

$$S(f) = \frac{P(f)}{\sigma^2} = \int_{-\infty}^{\infty} e^{-2\pi i f k}\,\rho(k)dk \tag{37}$$

4.1 Autoregressive (AR) Model

Notation of $AR(p)$ refers to the autoregressive model of order p. The $AR(p)$ model written as:

$$X_t = c + \sum_{i=1}^{p} \phi_i X_{t-i} + \varepsilon_t \tag{38}$$

Where $\phi_{1\,,\ldots,}\phi_p$ are autoregressive parameters, c is a constant, and the random variable ε_t is the white noise. Some constraints are necessary on values of parameters so that model remains stationary. For example, processes in $AR(1)$ model with $|\phi_1| \geq 1$ are not stationary. An autoregressive (AR) model is a description of a kind of random process in statistics and signal processing; as such, it represents such time-varying processes in nature, economics, etc.

4.2 Moving Average (MA) Model

Notation $MA(q)$ refers to the moving average model of order q:

$$X_t = \mu + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \varepsilon_t \tag{39}$$

Where the $\theta_1, \ldots \theta_n$ are moving average parameters of model, m is the expectation of X_t (often assumed to equal 0), and $\epsilon_t, \epsilon_{t-1}, \ldots$ are again, white noise error terms. A time series smoothed by moving average (MA), which can create cyclic and a trend like plots even when original data are independent random events with fixed mean. This role diminishes its usefulness as a control mechanism.

4.3 ARIMA again ARFIMA Model Time Series

The above AR and MA models generalized as follows:

$$\left(1 - \sum_{i=1}^{p} \phi_i B^i\right)(1 - B)^d (X_t - \mu) = \left(1 + \sum_{i=1}^{q} \theta_i B^i\right)\epsilon_t \tag{40}$$

The above $(1 - B)^d$ called a difference operator ∇^d. ARMA or ARIMA models can only catch the

SRD property, as d is restricted to the range of integer order. Therefore, the ARFIMA (p, d, q) model is proposed accordingly in order to capture the LRD property of the fractional systems. In fact, the operator can be defined in a natural way by using binomial expansion for any actual number d with Gamma function:

$$(1-\mathrm{B})^{\mathrm{d}}=\sum_{\mathrm{k}=0}^{\infty}\binom{\mathrm{d}}{\mathrm{k}}(-\mathrm{B})^{\mathrm{k}}=\sum_{\mathrm{k}=0}^{\infty}\frac{\Gamma(\mathrm{d}+1)}{\Gamma(\mathrm{k}+1)\Gamma(\mathrm{d}+1-\mathrm{k})}(-\mathrm{B})^{\mathrm{k}} \tag{41}$$

Many authors suggested that the use of a fractional ARIMA model by using a fractional difference operator rather than an integer could better consider this LRD phenomenon. In the general form of ARIMA (p, q, d) the Xt process in the above equation is defined as the ARFIMA (p, d, q) process:

$$\Phi(\mathrm{B})(1-\mathrm{B})^{\mathrm{d}}\mathrm{X}_{\mathrm{t}}=\Theta(\mathrm{B})\varepsilon_{\mathrm{t}} \tag{42}$$

where d $\epsilon(-0.5,0.5)$, and $(1-\mathrm{B})^{\mathrm{d}}$is defined as the fractional difference operator in ARFIMA(p, d, q) Processes are widely used in the modelling of the LRD time series, where p is a self-regressive order, q is the moving average order, and d is difference level. Greater value of d, closer simple integrated series approximated, and a general integrated series can be approximated better than mixed fractional difference and ARMA model. ARFIMA (p, d, q) process is natural generalization of the ARIMA or ARMA standard processes. In a fractionally differentiated model, the difference coefficient d is the first parameter to be calculated. Self-similar strength of ARFIMA can be calculated by parameter d. For finite variance cycle with fractional Gaussian noise, d has a closed relationship with Hurst parameter:

$$\mathrm{d}=\mathrm{H}-1/2 \tag{43}$$

In addition, for the infinite variance cycle with fractional a-stable noise, d is related to Hurst and characteristic exponent.

$$\mathrm{d}=\mathrm{H}-1/\alpha \tag{44}$$

In this way, parameter d can be chosen for modelling long-term effects, while p and q can be chosen for modelling relatively short-term effects. ACF sample and partial autocorrelation (PACF) function are useful qualitative tools for evaluating the presence of autocorrelation at individual lags. Ljung-Box Q-test is a quantitative way of testing self-correlation at multiple lags together. Ljung-Box test statistic is as follows:

$$Q(L)=N(N+2)\sum_{k=1}^{L}\frac{\hat{\rho}_k^2}{N-k'} \tag{45}$$

Where N is the sample size, L is the number of autocorrelation lags, and $\rho(k)$ is the sample autocorrelation at lag k. Under the null hypothesis, the asymptotic distribution of Q is chi-square with L degrees of freedom [12].

5 Simulation ARFIMA$(\mathbf{p},\mathbf{d},\mathbf{q})$ Models in R

We are now summarizing some of results for ARFIMA (p, d, q) model with an emphasis on estimating the differentiating parameter d. Consider the simple ARFIMA(p, d, q) model of the form.

$$\Phi(B)(1-B)^{d}X_t=\Theta(B)\epsilon_t \;\; for \;\; d\,\epsilon(-0.5,0.5) \tag{46}$$

Where $\{\epsilon_t\}$ is a white noise process with $E(\epsilon_t)=0$ and variance σ_ϵ^2 and B is the back-shift operator

such that $BX_t = X_{t-1}$. The polynomials $\Phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$ and

$$\Theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q \tag{47}$$

Have orders p and q with all their roots outside unit circle [32]. In this paper, it is assumed that $\{X_t\}$ is a linear process without a deterministic term. We now define $U_t = (1-B)^d X_t$, so that $\{U_t\}$ is an$ARMA(p,q)$, Process. The defined process is stationary and invertible (see (3)) and has a spectral density function. , $f_X(\omega)$, is given by

$$f_X(\omega) = f_U(\omega)(2\frac{sin\omega}{2})^{-2d} \quad \omega\epsilon[-\pi,\pi] \tag{48}$$

Where $f_U(\omega)$ is spectral density function of process$\{U_t\}$. Regression equation is a function $g(n)$ of sample size n where$g(n) = n^{\alpha} \quad 0 < \alpha < 1$. This regression estimator obtained by replacing the spectral density function with window smoothed period gram function

$$m = n^{\beta} \quad 0 < \beta < 1 \tag{49}$$

The third one is GPH estimator with mild modifications suggested by Robinson, denoted hereafter by $\hat{d}sp$. This estimator regresses $\{ln\, I(w_i)\}$ on $ln(2\, sin\, \omega_i/2))^2$ for $i = l, l+1, \ldots, g(n)$, where l is the lower truncation point which tends to infinity more slowly than $g(n)$. Robinson derived some asymptotic results for $\hat{d}pr$, when $d \in (-0.5, 0.5)$ and showed that this estimator is asymptotically less effective than a Gaussian estimator of maximum probability of d. Our bandwidth choice $g(n)$ is given by

$$g(n) = \begin{cases} A(d,\tau)n^{\frac{2\tau}{2\tau+1}} & 0 \le d \le 0.25 \\ A(d,\tau)n^{\frac{2\tau}{2\tau+1-2d}} & 0.25 \le d \le 0.5 \end{cases} \tag{50}$$

Where $A(d,\tau)$ has to be chosen appropriately [39]. This bandwidth cannot be calculated in practice, since it requires knowledge of true parameter d. However, this problem can be reverted by either replacing unknown parameter d in g (n) function with estimate. $\hat{d}p$ or $\hat{d}sp$. We use this g(n) since it satisfies conditions $\frac{g(n)}{n} \to 0$ and $g(n)\frac{\ln g(n)}{n} \to 0$ as g(n) and n go to infinity. The appropriate choice of optimal g(n) has been subject of estimator $\hat{d}_W$ is based on the period gram and it involves the function

$$Q(\xi) = \int_{-\pi}^{\pi} \frac{I(\omega)}{f_X(\omega,\xi)} d\omega \tag{51}$$

$$\mathcal{L}(\xi) = \frac{1}{2n}\sum_{j=1}^{n-1}\left\{\ln f_X(\omega_j,\xi) + \frac{I(\omega_j)}{f_X(\omega_j,\xi)}\right\} \tag{52}$$

Let $\{X_t\}$ be the process as defined and Then $U_t = (1-B)^d X_t$is an ARMA(p, q) Process and $y_t = \frac{\Phi(B)}{\Theta(B)}X_t$ is a ARFIMA(0, d, 0) process. Model Building Steps:

1. Estimate d in the ARIMA(p, d, q) model; denote the estimate by $\hat{d}$
2. Calculate $\hat{U}_t = (1-B)^{\hat{d}}X_t$
3. Using Box – Jenkins modelling procedure or the AIC criterion, identify and estimate ϕ and θ parameters in the ARMA(p, q) process $\phi(B)\hat{U}_t = \theta(B)\epsilon_t$

4. Calculate $\widehat{Y}_t = \frac{\hat{\phi}(B)}{\theta(B)} X_t$.

5. Estimate d in the ARFIMA$(0, d, 0)$ model$(1 - B)^{\hat{d}} \widehat{Y}_t = \epsilon_t$. The value of $\hat{d}$ obtained in this step is now the new estimate of d.

6 Concluding Remarks

In this paper, we simulate model ARFIMA (-0.4, 0.3, 0.1) by three times and each time it is generated with 2000 data, then model is fit again, and results can be checked as Table 1. We considered a simulation study to test the procedures for estimating parameters of an ARFIMA operation. Parametric approaches used in simulation and estimation by smoothed model. Results indicate that whether AR or MA with fractional components involve in parametric process. Our results provide strong evidence to support the existence of long memory in simulation data. This is inconsistent with weak market efficiency that can discuss in future papers. Simulation results for ARFIMA process represented in Table 2. Although we considered several values of parameter, the result of each iteration presented. As in the ARFIMA model, iterations of quantity need to achieve convergence, with latter requiring smallest. Results of estimators are very good. As mentioned in Table 1, initially with ARFIMA simulation has been done and again the assumed model has been fitted on it and according to the simulation values, the values of Table 1 have been extracted which is largely indicative for accuracy is too close to the real value. The autoregressive and moving average coefficients as well as the ARFIMA default time series fractal value is close to the real values.

Table 1: ARFIMA(z = sim, order = c(1, 0, 1))=(-0.4,0.3,0.1)

Coefficients for fits:		
	Coef.1	SE.1
ϕ = autoregressive parameter	-0.389461	0.0466055
θ = moving average parameter	0.0844606	0.0683559
D.F	0.279071	0.0330984
Fitted mean	0.0103125	0.129696
log	-5.48214	
σ^2	1.00659	

The output in table 2 corresponds to five different iterations of this simulation. Obviously, the relevant results can be discussed in future articles due to the existence of a deficit indicating the existence of long-term memory in the data and the inefficiency of the market accordingly. By new criterion in this paper, the deviation from the real value according to the output of Table 2 decreases in each iteration and the results will be convergent to the real value.

Table 2. Simulation Results

Simulation step	1	2	3	4	5
ϕ= autoregressive parameter	-0.4095738	-0.4764306	-0.3959723	-0.4523015	-0.3894609
θ = moving average parameter	0.1024239	0.02327982	0.027245	0.06180222	0.08446063
D.F	0.2935695	0.3029039	0.2443465	0.271685	0.2790712
σ^2	1.076657	1.007938	0.9940234	0.9664756	1.006587

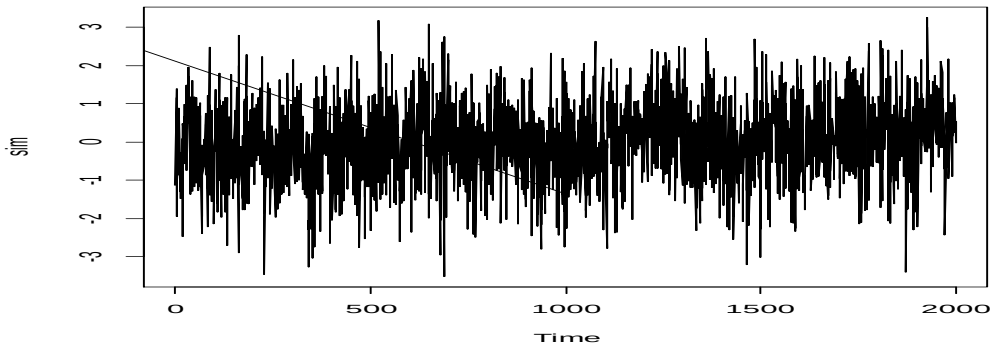

Fig. 1: ARFIMA Process Simulation

In Figure 1, the ARFIMA process simulated by fractal dimension value equal to 0.5, which indicates and show the data, occurred randomness state in them. In this figure, based on the accrue trend it is more chaotic statues in the data will be proportional to the time indicating that prove to more efficiency too.

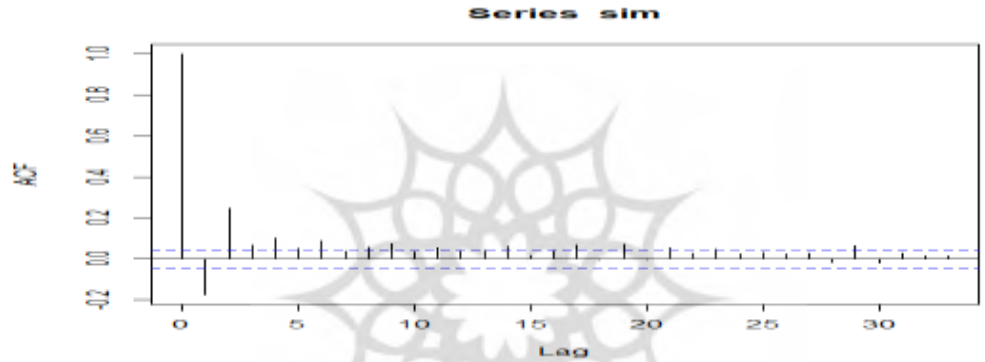

Fig. 2: ACF for ARFIMA Process Simulation

Figure 2 shows the autocorrelation of the data and state one, being in between two band, and its damping and descending diagram also has a good evidence of the random state of the data. This figure shows the accuracy of simulating the relevant adultery data.

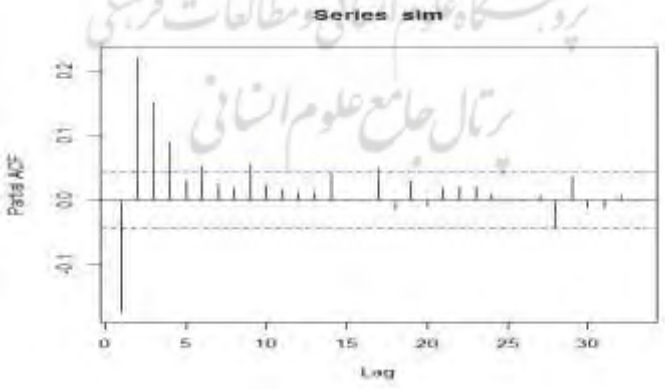

Fig. 3: PACF for ARFIMA Process Simulation

Figure 3, which is actually a partial autocorrelation diagram and used to determine the fit of the model on the simulated data, can also be used as a tool to determine the degree of randomness of the

data and estimated parameters of model to determine and the degree of autocorrelation was named. The different levels of this diagram and the band placement process also confirm the simulated fractal model. In this paper, we simulate fractal data using R software and then, knowing that simulated fractal data is known, approximate the Hurst dimensions using the mean method, and new method introduced in this paper for comparison. We calculate based on median and observe that the algorithm presented also it is closer to reality based on the median of the data.

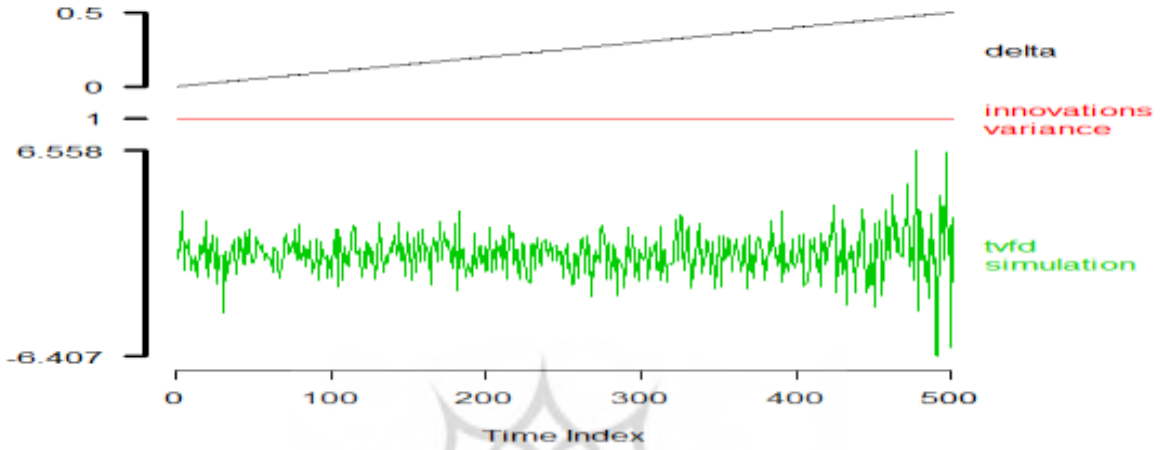

Fig. 4: Time Index for ARFIMA Process Simulation

Figure 4 also shows the changes in fractal dimensions and being close to 0.5. The degree of randomness of previously simulated data is also evident in certain duplications of simulated data. In the following, the accuracy of the fractal dimension has shown. The actual data simulated with a value of 0.5 and their estimation in the previous method of fractal dimension is equal to 0.537524, which obtained using the new criterion of 0.5015363. Simple R/S Hurst estimation with mean: 0.537524 and function for estimation fractal data with new method using by median is 0.5015363
The interpretation of the H values is the following:

1- If $0 < H < ½$ anti-persistent process.
2- If $H = ½$ a simple random process or ARMA process. There is a long-term dependence absence.
3- If $½ < H < 1$ long-term process, the dependence is even stronger as H tends towards 1.

We show that the variance of R/S Hurst estimation with mean is greater than the median from. In fact, by information about data can help us knowing about simulation that data is fractal and will have a greater impact on the accuracy its results. In this paper, we have shown that by using a change in the focus and substitution criteria instead of the average in the Hurst criterion, the accuracy of the fractal dimension estimator can increase, and this is important when interpreting the median substitution instead of the average and using it. As an indicator of focus, it will be doubly important due to the indifference to out-dated data caused by financial crises. The new Hurst criterion can use as a new tool in all of time series data and the output can be compared with previous results. Obviously, this article will be the beginning of new research in this field.

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