However, this mathematical language, as every other language, is far from being infallible:

> Rational reflection has therefore led us to conclude that also for pure mathematics there cannot be an infallible language, i.e., a language which in the exchange of thought excludes misunderstanding and in its mnemotechnical function provides a guarantee against error, i.e., confusing different mathematical entities. One cannot make provision against this inadequacy,... (Brouwer, 1990a)

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Refefences

Benecerraf, P; and H. Putnam. (ed.). (1983). *Philosophy of Mathematics: Selected readings. (2nd ed.)* Cambridge: Cambridge University Press.

Brouwer, L. E. J. (1981a). *On The Foundation of Mathematics.* PhD dissertation. In A. Heyting, (ed.) *Collected work.* North-Holland, Amsterdam.

Brouwer, L. E. J (1981b). "Historical background, principles and methods of intuitionism". In A. Heyting, (ed.) *Collected work.* North-Holland. Amsterdam.

Brouwer, L. E. J. (1981c) "Consciousness, Philosophy and Mathematics ". In *A.* Heyting, (ed.) *Collected work.* North-Holland, Amsterdam.

Brouwer, L. E. J. (1981d). Collected Works I, *Philosophy and Mathematics,* ed. A. Heyting, North-Holland, Amsterdam.

Brouwer, L. E. J. (1983). "Intuitionism and Formalism". In *Philosophy of Mathematics: Selected readings.* (2nd ed.) Cambridge University Press, Cambridge.

Brouwer, L. E. J. (1990a). "Volition, Knowledge and Language". In, *Brouwer's Intuitionism,* North-Holland, Amsterdam. appendix 5.

Brouwer, L. E. J. (1996). "Life, Art and Mysticism". Translate d by W. P. van Stigt, *Notre Dame Journal of Formal Logic,* 37(3): pp. 381-387.

Husserl, E. (1973). *Cartesian Meditation, An Introduction to Phenomenology.* translated by Dorion Cairns, Martinus Nijhoff, The Hague.

Kant, I. (1929). *Critique of Pure Reason.* tr. by Kemp-Smith, MacMillan, London.

Posy, C. (1998). "Brouwer versus Hilbert: 1907-1928." *Science in context,* 11(2): 291-325.

Van Atten, M. (2004). *On Brouwer.* Wasdworth Philosophical Series, Wadworth.

Van Stigt, W. P. (1990). *Brouwer's Intuitionism.* North-Holland, Amsterdam.

A natural question regarding the subjectivity of Brouwer's intuitionist construction of mathematics is the following: how can mathematics be *communicated* at all? Kant would have answered positively this question by resorting to his *pure forms of intuition*, which are "inherent" and the same in perception of all human beings. Human beings are so constituted as to have this cognitive structure, which is what makes communication *possible.*

Similar answer can be given on a Husserlian reading of Brouwer as well. In his transcendental phenomenology of "Ego", Husserl demonstrates *inter-subjectivity* as an essential property of the Subject. His rather long argument is based on the *unity* of consciousness, implicit in Cartesian *Cogito*, (Husserl, 1973).

Both interpretations of Brouwer's creating Subject, Kantian or Husserlian, make mathematical knowledge to be inter-subjective.

My suggestion is to *read Brouwer by himself*. Brouwer believes that

> Strictly speaking the construction of intuitive mathematics in itself is an *action* and not a *science*. (Brouwer, 1981a, p. 99)

As a *science*, mathematical activity, must be inter-subjective, in the first place. Brouwer's cornerstone for the inter-subjectivity of mathematical knowledge in his relatively long journey of Subject is:

> A very essential hypothesis in the mathematical viewing of fellow-man is the supposition of the presence in each of them of a mathematical-scientific mechanism of viewing, acting and reflecting similar to one's own. (Brouwer, 1990a)

The above hypothesis is a *necessary* condition for mathematics to be the same for all subjects. Here is the place where other philosophies come on the scene to *interpret* or *derive* the above *hypothesis* in terms of other hypotheses. For example, in Kantian philosophy, it is derived, by a transcendental argument, from the *hypothesis* of the *possibility of knowing.* In Husserlian phenomenology, it is *reduced* to a phenomenological *interpretation* of Subject. My suggestion is to *accept* it as a *hypothesis*.

These subjects should also have a *means* to share this knowledge with one another. That *means* is a mathematical *language*:

> ...words which they would use only as invariant symbols for definite elements and relations between elements of pure mathematical systems created by them. (Brouwer, 1990a)

3. Ego and Inter-subjectivity

The notion of *Ego* or *Subject* is a central notion in Brouwer's philosophy in general and his philosophy of mathematics, in particular. In his writings on this crucial matter, Brouwer never refers to Kant. However, there is a debate as to whether we should *interpret* Brouwer's concept of Subject in the Kantian *transcendental subject*'s framework or within the Husserlian *transcendental phenomenology*. For a Kantian interpretation of Brouwer's notion of Subject, see, e.g., (Posy, 1998) and for a Husserlian one, see (Van Atten, 2004). Historically speaking, Brouwer was familiar with Kant, when he wrote his dissertation, and with Husserl around at least 1928, (Van Atten, 2004).

Brouwer's concept of the *idealized* Subject or *creating* Subject is essentially based on the existence of "I", the "Self", and the *individual* nature of mathematical activity. Combining these facts with Brouwer's emphasis on the *essentially languageless* nature of mathematical activity yield to the "Solitude" of the individual Subject.

> In a human mind equipped with an unlimited power of memory pure mathematics, practiced in solitude and without the use of symbols, would be exact. (Brouwer, 1990a p. 58)

According to Brouwer, there is no *objectivity* for mathematics, since the concept of objectivity itself presupposes the existence of *objects* external to the mind, that belongs to *second order* mathematics.

> ...They belong to what in the next chapter we shall call mathematics of the second order. (Brouwer, 1981a, p. 119)

It should be noted that Brouwer's concept of the "second order" here is *totally* different from what is called the "second order" nowadays, usually attributed to logic. From Brouwer's point of view, the first order mathematics is the *real* and *exact* mathematics *constructed* purely in the mind, before to be dressed with any language. On the other hand, the characterization of logics in terms of *order* is related directly to the types of quantifiers in propositions.

So according to Brouwer, mathematics, as an a priori knowledge cannot be objective. However, Brouwer is persistently concerned with *truth* and *certainty* of mathematics. Both of these characters come from the nature of mathematical activity.

M. van Atten rightly claims (2004), that the *function* of the intuition of time with respect to mathematics is the same for Kant and Brouwer, although the *way* this intuition comes about is different.

2. Synthetic a priority of mathematics

It is well known that in his basic classification of statements, Kant classifies mathematical propositions as *synthetic a priori judgements* (Kant, 1929, p. 757). Kant argues that mathematical theorems are proved by axioms *and* the law of contradiction. That means that the negation of mathematical propositions are not *self*-contradictory, so they are *not* analytic, (p. 14). According to Kant, mathematical axioms are synthetic since they are not established by the analysis of a given concept, but only by *intuitive construction* of the concept, which will show the necessary presence of attributes not included in a logical definition of the subject, (Husserl, 1973, p. 758).

Brouwer admits the synthetic a priority of mathematics in (Brouwer, 1981a). He accepts the synthetical character of mathematical propositions, since besides other reasons, he *denies* an analytic proposition having any *meaning*. Brouwer does *not* recognize "truth-in-themselves", (1990a).

He says:

> Truth is only in reality, i.e., in the present and past experiences of consciousness ... There are no non-experienced truths. (1990a)

To experience a truth is to experience that a certain construction has succeeded. Passing through, it should be mentioned that Brouwer's notion of "experiencing" truth, does not have any connection to *outer* experience, but is mind's *internal* experience.

For Brouwer, mathematical knowledge is a priori, since

> Mathematics is a free creation of the mind independent of experience, it develops from a single a priori intuition, (Brouwer, 1981a).

So Brouwer and Kant, both believe in the synthetic a priority of mathematics, but they build their arguments on *different* basis.

is divested of all quality, there remains the empty form of the common subsratum of all two-ities. It is this common substratum, this empty form, which is the basic intuition of mathematics, (Brouwer,). Brouwer points out that to have a two-ity as a starting point is *necessary*, and unity is *not* sufficient, (Brouwer, 1981a). His reason that the unity is not enough is based on the presupposition of *addition* in every way of constructing numbers from unity. What Brouwer calls the "two-ity" is the pure form of the experience "then-now". The "then" corresponds to 1 and the "now" to 2, (Van Atten, 2004). But as time passes, the second element of the two-ity itself falls apart into two. In this way, we get 3, and continuing this way, we get the other natural numbers, (Brouwer, 1990a).

Brouwer believes that

> However weak the position seemed to be after this period of mathematical development, it has recovered by abandoning Kant's a priority of space but adhering the more resolutely to the a priority of time. (1983a, pp. 11-12)

So he *explicitly* admits his acceptance of the *a priority* of time from Kant. Now the question is whether his notion of "intuition of time" is exactly the same as Kant's.

In his genealogy of *consciousness*, (1990a), Brouwer recognizes a *moment* where the intuition of time *happens* for the Subject. This moment of the Subject is the stage where consciousness passes from its *deepest home*, where it seems to oscillate slowly, will-lessly, and reversibly between stillness and sensation. It is the only status of sensation that allows the *initial phenomenon* of that transition. This initial phenomenon is a *move of time*. The awareness of the move of time is what makes consciousness as *mind*. In contrast to Brouwer's notion of the intuition of time, where consciousness *actively becomes* mind, Kant's *a priori* intuition of time is *passive*:

> ... a pure intuition, which even without any actual object of the senses or of sensation, exists in the mind a priori as a mere form of sensibility. (Kant, 1929, p. 35)

So Kant's intuition of time has a firm basis, mind's structure, which makes every experience possible. That is *not* something which *is born* in the life process of mind.

Brouwer presents a second argument against [only] the *conclusion* of Kant's argument by raising the question: cannot the human intellect be organized [as well] in such a way that it can place the conception of an external world in other receptacles, without this being realized *in practice*; for instance because it has little effect and consequently our capacity for it receives very little practice? (Brouwer, 1981a, pp. 115-6).

Although Brouwer does not accept *a priority* of space, he believes in the *a priority* of time. He begins chapter II of his dissertation with:

> Proper to man is a faculty which accompanies all his interaction with nature, namely the faculty of taking a *mathematical view* of life, of observing in the world repetition of sequences of events, i.e., of causal systems in time. The basic phenomenon therein is the simple intuition of time, in which repetition is possible in the form: "thing in time and again thing", as a consequence of which moments of life break up into sequences of things which differ qualitatively. These sequences thereupon concentrate in the intellect into mathematical sequences, not *sensed*, but *observed*. (p. 81)

And later in the same chapter, he adds:

> ... we can call *a priori* only that one thing which is common to all mathematics and is on the other hand sufficient to build up all mathematics, namely the intuition of the many-oneness, the basic intuition of mathematics. And since in this intuition we become conscious of time as change *per se*, we can state: *the only a priori element in science is time.* (p. 99)

In a footnote to the above text, Brouwer distinguishes the *intuition* of time from what he calls *scientific* time. He argues that the latter is an *a posteriori* concept, since it only appears in experience and as a one-dimensional coordinate with a one-parameter group structure that can be introduced for cataloguing of phenomena.

By this intuition of time, or *the move of time*, Brouwer builds his mathematics. He then describes how the subject creates the numbers 1 and 2, simultaneously. Intuitionist mathematics is an *essentially languageless* activity of the mind which has its origin in the perception of the move of time, i.e., *the falling apart of a life moment into two distinct things*, one of which gives way to the other, but is retained by memory. If the *two-ity* thus born

(3) inter-subjectivity of mathematical constructions.

The notion of *the movement of time* as an *a priori intuition of time*, as explicitly expressed by Brouwer, is more or less from Kant. However, we will argue that their notions of "intuition" are not the same.

About the second item, Brouwer believes that *all* of mathematical knowledge is a priori and synthetic. His arguments are different from Kant's arguments for these notions.

The concept of "inter-subjectivity" in mathematics in Brouwer's philosophy is very involved, and there is no reference to Kant. One may *interpret* it by Kantian *transcendental subject*, Husserlian *transcendental phenomenology*, or still other ways. We will briefly mention the possible Kantian and Husserlian interpretations. My suggestion is to *read Brouwer by himself*.

1. Intuition of Time

In chapter II of his PhD dissertation, *Mathematics and Experiences*, (Brouwer, 1981a, pp. 114-5), Brouwer considers Kant's notions of space and time. He argues that Kant's argument for *a priority* of space is *wrong*. To show his claim, Brouwer argues that Kant proves his statement "the perception of an external world by means of a three-dimensional Euclidean space is an invariable of the human intellect; another property of an external world for the same human beings is a contradictory assumption", (Kant, 1929, pp. 114-5) by decomposing it into:

(1) We obtain no external experience barring those placed in empirical space, and cannot imagine those experiences apart from empirical space,

(2) For empirical space the three-dimensional Euclidean geometry is valid.

So it follows that the three-dimensional Euclidean geometry is a necessary condition for all external experiences and the only possible receptacle for the conception of an external world so that the properties of Euclidean geometry must be called *synthetic judgments a priori* for all external experience.

Brouwer argues that we may *obtain* our experiences apart from all of mathematics, hence apart from any space conception. He then claims that the creation of a space conception is a *free action* of the intellect. He concludes that the first assumption is definitely false and therefore, the conclusion that three-dimensional Euclidean geometry is *a priori*, must also be rejected.

contribution to the ongoing debate on the foundations of mathematics. His dissertation revealed the twin interests in mathematics that dominated his entire carrier, his fundamental concern with critically assessing the foundations of mathematics, which led him to his creation of *intuitionism,* and his deep interest in geometry, which led him to his seminal work in topology. Brouwer is most famous for his contribution to the philosophy of mathematics and his attempt to build up mathematics anew on an intuitionistic foundation, in order to meet his own searching criticism of hitherto unquestioned assumptions. Brouwer was able to step outside the established mathematical tradition. His questioning of principles of thought led him to a revolution in the domain of logic.

Brouwer characterized mathematics primarily as the free activity of exact thinking. This activity is founded on the *pure intuition of time*. In his philosophy of mathematics, there is no independent realm of objects and language does not play a fundamental role. So he avoids Platonism with its epistemological problems and Formalism with its poverty of contents. On Brouwer's view, there is no mathematical truth outside the activity of thinking. Then a proposition becomes true only when the subject has experienced its truth, that means having an appropriate mental construction. Similarly, a proposition is false only when the subject experiences its falsehood, that means when the subject realizes that an appropriate mental construction is not possible. By Brouwerian reading of the logical connectives, *tertium non datour* is not a valid logical axiom scheme, since as he claims "there is no non-experienced truth". Brouwer followed his philosophy of mind building his mathematics. His intuitionist arithmetic is a subsystem of classical arithmetic and his intuitionistic analysis is incompatible with classical analysis.

It is a well-known fact in intuitionist school of mathematics that Immanuel Kant (1724-1804) had a *direct* influence on Brouwer: **In Kant we find an old form of Intuitionism.** (Brouwer, ?, p. 3)

What is debatable is the *extent* of the influence, and recently, it has been argued that to have a better understanding of Brouwer, we should *revise* our old beliefs, (Van Atten, 2004).

In this paper, we review the domains of Brouwer's philosophy of mathematics where Kant may had explicit and implicit influence.

There are at least three elemental parts in Brouwer's philosophy of mathematics that may have their origin in Kant. These three parts are:

(1) the intuition of time,

(2) the synthetic a priority of mathematical knowledge, and

Kant's Influence on Brouwer

Mohammad Ardeshir[*]

Abstract

There are at least three elemental parts in Brouwer's philosophy of mathematics that may have their origin in Kant. These three parts are (1) the intuition of time, (2) the synthetic a priority of mathematical knowledge, and (3) the inter-subjectivity of mathematical constructions. Brouwer borrowed the notion of the movement of time as an a priori intuition of time, explicitly expressed, from Kant. In Brouwer's philosophy of mathematics, the intuition of time is the only a priori notion, on which the whole of mathematics is built. However, their notions of the "intuition of time" are not the same in the genealogy of mind.

As far as the second item is concerned, Brouwer believes that all of mathematical knowledge is a priori and synthetic. His arguments are different from Kant's arguments.

The concept of "inter-subjectivity" of mathematics in Brouwer's philosophy is very involved, and there is no reference to Kant in this respect. One may interpret it by the Kantian transcendental subject or even the Husserlian transcendental phenomenology. Both interpretations seem to be consistent. My suggestion is to read Brouwer by himself.

***Keywords**: Brouwer, Kant, intuition, inter-subjectivity, language, mathematics, philosophy, subject.*

* * *

0. Introduction

From an early stage L. E. J. Brouwer (1881-1966) was interested in the philosophy of mathematics, mysticism and other philosophical questions relating to human society. He published his own ideas on this topic in 1905 in his treatise *Leven, Kunst en Mystiek* (Brouwer, 1996, p. 2). Brouwer's doctoral dissertation, published in 1907, (p. 3), made a major

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