Risk Measurement and Implied Volatility Under Minimal Entropy Martingale Measure for Levy Process

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ARTICLE INFO	ABSTRACT

This paper focuses on two main issues that are based on two important concepts: exponential Levy process and minimal entropy martingale measure. First, we intend to obtain risk measurement such as value-at-risk (VaR) and conditional value-at-risk (CvaR) using Monte-Carlo method under minimal entropy martingale measure (MEMM) for exponential Levy process. This Martingale measure is used for the exponential type of the processes such as exponential Levy process. Also, it can be said MEMM is a kind of important sampling method where the probability measure with minimal relative entropy replaces the main probability. Then we are going to obtain VaR and CVaR by Monte-Carlo simulation. For this purpose, we have to calculate option price, implied volatility and returns under MEMM and then obtain risk measurement by proposed algorithm. Finally, this model is simulated for exponential variance gamma process. Next, we intend to develop two theorems for implied volatility under minimal entropy martingale measure by examining the conditions. These theorems consider the asymptotic implied volatility for the case that time to maturity tends to zero and infinity.

1 Introduction

In this paper, we pursue two goals that are based on two important and attractive concepts. These two purpose are considering and computing the implied volatility and risk measurement and those two important concepts are Levy process and minimal entropy martingale measure. The first considering concept is Value-at-Risk. Value-at-Risk (VaR) has become a significant measure for approximating and controlling portfolio risk introduced and expressed by Jorion in 1997 and Wilson in 1999. VaR interpreted as a certain quantile of the change in a values of the portfolio in the determined interval. For more explanation, let the current value of the portfolio is $V(t)$, the determined time interval is Δt, and the portfolio value at time $t + \Delta t$ is $V(t + \Delta t)$. The loss in portfolio value in the determined interval of the time is $L = -\Delta V$ where $\Delta V = [V(t + \Delta t) - V(t)]$ and the VaR, x_p, related with a specified probability p is determined by

$$P\{L > x_p\} = p \tag{1}$$

i.e., the VaR x_p, is the $(1 - p)$'th quantile of the loss distribution. To estimate (1), Monte Carlo simulation is often applied; variations in the portfolio's risk factors are implemented, the portfolio is analyzed, and the loss distribution is obtained. In [8] has been described and evaluated variance reduction techniques for efficient estimation of portfolio loss probabilities using Monte-Carlo simulation to generate changes in risk factors. In [17] can be seen the application of VaR and CVaR

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for selecting the best input for the proposed model. In [7] has been proposed, analyzed and approximated an algorithm for estimating portfolio loss probabilities using Monte-Carlo simulation. Gaining careful assessment of such loss probabilities is fundamental to computing VaR. The methods have been reviewed in [9] try to incorporate the best features of two procedures to computing VaR: reduce time of the delta-gamma estimation and the precision of Monte-Carlo simulation. The next concept that has been considered is implied volatility. Volatility measures variations and changes of the price in a determined interval. The volatility actually determines price stability, a stock with low volatility has a more stable price and a stock with high volatility has large variations in price. Implied volatility is often known as predicted volatility often referred to as projected volatility, indeed implied volatility presented an approximation for stock volatility in the future using option prices. It should be noted that finding implied volatility is not based on accurate calculations. In fact, implied volatility is a forward-looking computation. Unlike implied volatility, historical volatility is backward-looking and is computed by the variety of prices previously specified. Historical volatility does not discuss market direction, rather, it regards at how far a price deviates from its average value, up or down, in the determined period. One of the issues that that matters to us is considering the behavior of volatility surface related to the values of strike price and time to maturity. The asymptotic behavior of the properties for exponential Levy process have been considered and focused with the large and small strike behavior and studied the short maturity asymptotic, where it turns out that the behavior of the implied volatility is very different for out-of-the-money (OTM) and at-the-money (ATM) options. Miyahara and Moriwaki in [15] after reviewing the structure of MEMM for exponential Levy process, they have studied the properties of volatility smile/smirk for the model. Tehranchi has considered the implied volatility behavior for asymptotic states and the results can be seen in [21,22]. In [22] has been obtained the key results for implied volatility surface for large times to maturity. The variations of implied volatility for the state of far from maturity has been discussed in [21]. As we know, a pricing model includes two main sections: the first sections are a pricing process of assets selections and the second sections is selection of suitable method for computing the option pricing. Here the exponential Levy model chooses for the price process and minimal entropy martingale measure chooses for computing the option pricing. In the following, we will discuss the reasons for the choice and the advantages of these two concepts. In the late 1980s and early 1990s, Lévy models were proposed as an alternative to improve on the consequence of the Black-Scholes model. A method for solving Black-Scholes model is proposed in [18]. A great gain of using Levy models is that they obtain account the stylized specifications of the markets. Numerical Solution of Multidimensional Exponential Levy Equation can be seen in [1]. If Black-Scholes model is allowed to jump the asset price, while the independent and stationary of returns is maintained, then we will have exponential Levy model. There are many reasons to use the jumps in financial modeling. For example, asset prices do jump, and continuous-path models cannot cover this risk well [11]. In continuous-path models, the probability that the asset price will move by a large amount over a short period of time is exceptionally small, unless an unrealistically high value of the volatility is established. As another explanation for using jump models proposed, it can be seen smile situation in the implied volatility chart of the option market, which implies that the risk-neutral returns are non-Gaussian and leptokurtic. The presence of very prominent and clear smile for short maturity is one reason for existence jump in the option market. The fact that there are in jump model as well as real market is that the law of return is less close Gaussian in short maturity, whereas in continuous-path models, the law of returns is closer to Gaussian. It can also be said that continuous-path models are used to model complete markets and jump model to model incomplete markets. The next issue that is important in risk management is that the jump models let investors quantify and take into account the risk of large asset price movements over short time intervals, something that is absent in continuous-path models.

On the other hand, various suggestions for an appropriate equivalent martingale measure are offered. That can be mentioned as: Minimal Martingale Measure (MMM), Variance Optimal Martingale Measure (VOMM), Mean Correcting Martingale Measure (MCMM), Esscher Martingale Measure (ESSMM), Minimal Entropy Martingale Measure (MEMM) and Utility-Based Martingale Measure (U-MM). We will focus on minimal entropy martingale measure for the following advantages:

1) Relationship to Kullback-Leibler Information: In the MEMM, the distance calculation method is Kullback-Leibler, therefore, it can be said this martingale measure has minimal distance in the sense of Kullback-Leibler to the original probability P.

2) Relationship to Large Deviation Theory: it can be said MEMM is the most possible empirical probability measure of paths of price process in the class of all equivalent martingale measures based on Sanov theorem.

3) Relationship to Esscher Transformation: MEMM can be obtained using Esscher transform and the Esscher transform is very common in the risk management.

4) Appropriate for applying in exponential Stable Model: minimal entropy martingale measure is the only one martingale measure which can be applied to the exponential stable process model.

5) Relationship to exponential utility function and indifference price: This is important economically and financially.

6) Smile/Smirk Property of implied volatility surface: The property is an important feature that a good pricing model should have.

The equivalent martingale measure (EMM) for exponential Levy model has been considered by different researchers. In [5], have been proven that under an exactly smooth condition, there is the minimal entropy martingale measure (MEMM) for the exponential Levy model. In that paper was attained a EMM called the minimal entropy martingale measure which minimizes the relative entropy between equivalent martingale measure and the main probability measure. Delbaen et. al. in [4] have considered the duality relation between the minimizes the relative entropy which specifies the MEMM and the maximal expected exponential utility. In [12], the f^q-minimal martingale measure has been determined as equivalent local martingale measure which minimizes the f^q-divergence $E[(\frac{dQ}{dP})^q]$. In [10] have been introduced some new results on the issue that the equivalent Esscher martingale measure can be applied for the linear processes and exponential processes, and the minimal entropy martingale measure can be applied for exponential Levy process. Here are some articles that have used the Monte-Carlo and quasi Monte-Carlo method for option pricing. These methods have one major difference, and that is the employing of sequence generators. Pseudorandom number generators employed in Monte-Carlo method while quasi Monte-Carlo using low-discrepancy sequence whose distribution is much more even. For low dimensions, using Quasi Monte-Carlo is a better choice. However, pseudorandom number have a random appearance, but nevertheless exhibiting a specific, repeatable pattern. The low-discrepancy sequence is much more even in distribution than pseudorandom numbers. Fujisaki and Zhang have presented an effective procedure to specify approximately European option pricing and option pricing of Asian for compound Poisson process, mixture model and stable process under minimal entropy martingale measure using Monte-Carlo method in [6]. In [23] can be seen the preliminary presentation of the Monte-Carlo implementation for the American option pricing. But, quasi Monte-Carlo method displays significant outcome compared to the Monte-Carlo method. In [14] have been used the methods of Monte-Carlo and quasi Monte-

Carlo for European option pricing. In [20] has been concentrated on calculating the value to the minimal relative entropy between the main probability with respect to all of the equivalent martingale measure for the Levy model by Monte-Carlo and quasi Monte-Carlo using low-discrepancy sequence. The issues raised in this article briefly as follows. At the beginning, the primary concepts are explained. Such as exponential Levy process, equivalent martingale measure, implied volatility and risk measurement. In the next section, discussion and results are expressed. In the section we first describe two issues: computing the option price and implied volatility under MEMM for Levy process and risk measurement under MEMM. Then give a numerical example to explain the procedure. In this example, we first simulate a variance gamma process and then calculate option price, implied volatility and returns. In the following, we obtained VaR and CVaR using Monte-Carlo method. Finally, the methods for consider implied volatility and theorems presented short maturity asymptotic for at the money option and flattening of the smile far from maturity for Levy process under MEMM.

2 Preliminaries

The primary concepts are stated in the following. First, the exponential Levy process and in particular variance gamma process is explained. Then the equivalent martingale measure especially minimal entropy martingale measure is expressed. In the following, the implied volatility and the risk measurement such as VaR and conditional VaR are stated.

2.1 Levy Process

Suppose that $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_t; 0 \geq t \geq T)$. (the following concepts can be seen in [16].

Definition 2.1.1. A stochastic process $(X_t)_{t\in[0.T]}$ with $X_0 = 0$ almost surly is a Levy process if it is a cadlag, adapted, real valued stochastic process and possesses the following properties:

- Independent increments: $X_t - X_s$ is independence of $\mathcal{F}_s$ for any $0 \leq s < t \leq T$.

- Stationary increments: for any $0 \leq s. t \leq T$ the distribution of $X_{t+s} - X_t$ does not depend on t.

- X is continuous in probability: for every $0 \leq t \leq T$ and $\epsilon > 0, \lim_{s \to t} P(| X_t - X_s | > \epsilon) = 0$.

Definition 2.1.2. A levy process X_t with generating triplet $(\sigma^2. v(dx). b)$ has the next display which is correspond to levy-Ito decomposition:

$$X_t = \sigma W_t + bt + \int_0^t \int_{|x|>1} x N_p(dudx) + \int_0^t \int_{|x|\leq1} x \widetilde{N}_p(dudx)$$

$$(2)$$

W_t in the above formula is a Wiener process, $N_p(dudx)$ is a Poisson random measure and $\widetilde{N}_p(dudx) = N_p(dudx) - \widehat{N}_p(dudx)$ that $\widehat{N}_p(dudx) = duv(dx)$ is the compensator.

Definition 2.1.3. The Levy-Khintchine expression presents the characteristic function of X_t with generating triplet (σ^2, v, b) under probability P, which is displayed the correspondence between Levy models and infinitely divisible distributions.

$$\Phi_t(u) := \mathbb{E}[e^{iuX_t}] = e^{t\Psi(u)}, u \in \mathbb{R}$$

$$(3)$$

here Ψ named the characteristic exponent and it is determined by

$$\Psi(u) = ibu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux1_{|x|\leq 1}\right)\nu(dx) \tag{4}$$

$\nu(dx)$ is a measure on $(-\infty, \infty)$, where:

$$\nu(\{0\}) = 0 \quad and \int_{|x|>0}(|x|^2 \wedge 1)\nu(dx) < \infty \tag{5}$$

Definition 2.1.4. An exponential Levy model S_t has two kinds of representation which defined by $S_t = S_0 e^{X_t} = S_0 \mathcal{E}(\hat{X})_t$, where $S_0 > 0$ is a constant and $(X_t)_{t\in[0,T]}$ is a Levy process with generating triplet $(\sigma^2, \nu(dx), b)$ and $(S_t)_{t\in[0.T]}$ is the solution to the next equation:

$$dS_t = S_{t-}d\hat{X}_t \tag{6}$$

Definition 2.1.5. An exponential levy model S_t has another display of the form of $S_t = S_0 \mathcal{E}(\hat{X})_t$ that $(S_t)_{t\in[0.T]}$ is the solution to the next relation:

$$dS_t = S_{t-}d\hat{X}_t \tag{7}$$
and

$$\hat{X}_t := X_t + \frac{1}{2} < X^c >_t + \sum_{s\in(0.t]} \{e^{\Delta X_s} - 1 - \Delta X_s\} \tag{8}$$

the process $(X_t^c)_{t\in[0.T]}$ is the continuous part of $(X_t)_{t\in[0.T]}$. $\mathcal{E}(\hat{X})_t$ is called the Doleans-Dade exponential of $(\hat{X}_t)$. $(\hat{X}_t)_{t\in[0.T]}$ is still a levy process and its generating triplet is given by $(\sigma^2. \hat{\nu}(dx). \hat{b})$ that

$$\hat{b} = b + \frac{1}{2}\sigma^2 + \int_{|x|\leq 1} x \,\hat{\nu}(dx) - \int_{|x|\leq 1} x\nu(dx) \tag{9}$$

$$\hat{\nu}(x) = \frac{1}{1+x}\nu(log(1+x)) \tag{10}$$

There are various Levy models in terms of finite and infinite activity, variance gamma process which has an infinite activity is used here.

Definition 2.1.6. The variance gamma process with three parameters σ, ν, and θ is shown by $X_{VG}(t; \sigma, \nu, \theta)$ and determined as follows:

$$X_{VG}(t; \sigma, \nu, \theta) = \theta G(t; \nu) + \sigma W[G(t; \nu)] \tag{11}$$

where $W(t)$ be an standard Brownian motion, and $G(t; \nu)$ be an independent gamma process with unity mean and variance rate ν. The characteristic function of the variance gamma process is stated in the following: (see [2]):

$$\phi_{VG}(u,t) = E\{\exp[iuX_{VG}(t)]\} = (\frac{1}{1 - i\theta vu + \sigma^2 vu^2/2})^{t/v} \tag{12}$$

The variance gamma process has a Levy measure like the following: (see [13]):

$$v_{VG}(x) = \frac{1}{v|x|} e^{\frac{\theta}{\sigma^2}x - \frac{\sqrt{\theta^2 + 2\sigma^2/v}}{\sigma^2}|x|}. \tag{13}$$

2.2 Equivalent Martingale Measure

The equivalent martingale measure or the risk-neutral measure is main instrument for option pricing. Suppose the market is incomplete and without arbitrage, in that case there are several equivalent martingale measures. Therefore, the question of the most appropriate choice for martingale measure in order to option pricing is arisen. (see the definitions in [16]).

Definition 2.2.1. A probability $\mathbb{Q}$ defined on $(\Omega.\mathcal{F}.\mathbb{P})$ will be named an equivalent martingale measure for S_t if the process $e^{-rt}S_t$ is $\{\mathcal{F}_t\}$-martingale and $\mathbb{Q}\sim\mathbb{P}$.

Definition 2.2.2. Let $(X_t)_{t\in[0.T]}$ be a levy process on probability space $(\Omega.\mathcal{F}.\mathbb{P})$. The Esscher equivalent martingale measure $\mathbb{P}^*$ is constructed by Esscher transform for $(X_t)_{t\in[0.T]}$ by a density process Z_t of the following form:

$$Z_t = \frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t} = \frac{e^{\theta X_t}}{\mathbb{E}[e^{\theta X_t}]} \quad .\theta \in \mathbb{R} \tag{14}$$

Definition 2.2.3. A probability $\mathbb{Q}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is named an equivalent martingale measure for S_t if the process $e^{-rt}S_t$ is $\{\mathcal{F}_t\}$-martingale and $\mathbb{Q}\sim\mathbb{P}$ ($\mathbb{Q}$ and $\mathbb{P}$ are equivalent probability measure).

$$\forall\mathbb{Q}: equivalent\ martingale\ measure\ \ H(\mathbb{P}^*|\mathbb{P}) \leq H(\mathbb{Q}|\mathbb{P})\ ; \tag{15}$$

where;

$$H(\mathbb{Q}|\mathbb{P}) = \begin{cases} \int_{\Omega} log\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] d\mathbb{Q} \ ; if\ \mathbb{Q} \ll \mathbb{P} \\ \infty \qquad\qquad ;\ otherwise \end{cases} \tag{16}$$

Then, $\mathbb{P}^*$will be named minimal entropy martingale measure (MEMM) of S_t.

Theorem 2.2.1. (see [16]) Suppose that, there exist $\gamma^* \in \mathbb{R}$ which is the outcome of the equation (18) and satisfies in the phrase (17):

$$\int_{\{x>1\}} e^x e^{\gamma^*(e^x-1)}v(dx) < \infty \tag{17}$$

$$\Phi(\gamma^*) = b + \left(\frac{1}{2} + \gamma^*\right)\sigma^2 + \int_{\{|x|>1\}} (e^x - 1)e^{\gamma^*(e^x-1)}v(dx)$$

$$+ \int_{\{|x| \leq 1\}} ((e^x - 1)e^{\gamma^*(e^x - 1)} - x)\nu(dx) = r \tag{18}$$

and let $\mathbb{P}^*$ defined by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t} = \frac{e^{\gamma^* X_t}}{\mathbb{E}[e^{\gamma^* X_t}]} = e^{\gamma^* X_t + \Psi(\gamma^*)T} \tag{19}$$

Then the probability measure $\mathbb{P}^*$ is the MEMM of S_t and Z_t is a Levy process with respect to $\mathbb{P}^*$, and it's generating triplet (A^*, ν^*, b^*) under this probability as defined below :

$$A^* = \sigma^2 \tag{20}$$

$$\nu^*(dx) = e^{\gamma^*(e^x - 1)}\nu(dx) \tag{21}$$

$$b^* = b + \gamma^* \sigma^2 + \int_{\mathbb{R}\setminus\{0\}} x 1_{\{|x| \leq 1\}}(x)d(\nu^* - \nu) \tag{22}$$

2.3 Option Pricing and Implied Volatility

Let $S_T = S_0 e^{Z_T}$ be the final price of the asset of a European call option with strike K, where $(Z_t)_{t \in [0,T]}$ is a Levy process with triplet (b, σ^2, ν). $\mathbb{P}^*$ is chosen by minimal equivalent martingale measure method. Suppose that the log of the normalized strike is $k = \log(\frac{K}{S_0})$. The equivalent martingale measure valuation under $\mathbb{P}^*$ yields

$$C(K) = e^{-rT}E^*[(S_T - K)_+] = S_0 e^{-rT}E^*[(e^{X_T} - e^k)_+] \tag{23}$$

Let the price of the European call option for the determined model with the strike K, given by $C(K)$ as stated above, then the value of σ satisfies the equation:

$$S_0 N(d_1) - e^{-rT} K N(d_2) = C(K) \tag{24}$$

the variable σ can be obtained from $N(d_1)$ and $N(d_2)$ with

$$d_{1,2} = \frac{\log\frac{S_0}{K} + \left(r \pm \frac{\sigma^2}{2}\right)T}{\sqrt{\sigma}\sqrt{T}} \tag{25}$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

(26)

Then σ, defined as the implied volatility with respect to a determined model. The implied volatility is studied as a function of K and the implied volatility can have the smile or smirk property.

2.4 Risk Measurement

Two popular indexes to quantifying the risk are the Value-at-Risk and the Conditional Value-at-Risk. The VaR of the payoff is the maximum possible value of the payoff that can happen with a certain confidence level. briefly,

$$VaR_\alpha = \min\{z | F_X(z) \geq \alpha\} \quad for \; \alpha \in (0,1)$$

(27)

where α is the certain level of confidence. As the value-at-risk only measures and detects the level of risk exposure, it was completed by the CVaR which tends to lead to a more conservative approach in risk exposure and is defined by:

$$CVaR_a(H) = \frac{1}{1-a} \int_a^1 VaR_x(H)dx = E[H|H > VaR_a(H)]$$

(28)

There are three ways to estimating VAR: the historical method, the variance-covariance method, and the Monte-Carlo simulation. Historical Simulation method specified as "nonparametric VaR" is the distribution of cost and benefit limited area at a given confidence level using studying the results of backdated data on the existing portfolio. The idea on which the VaR calculation is using Histirical Simulation method is explained that by distributing the historical returns of the securities below the portfolio, you simulate VaR by assuming we have kept the current portfolio from the beginning of historical data. Another popularly method that is defined as parametric method is Variance-Covariance. In order to computing the volatility and correlations of the portfolio returns, this method is based on Variance-Covariance matrix of portfolio returns by use of the historical time series. Another method suggested by Banking Regulation and Supervision Agency (BRSA) in 2006, is Monte-Carlo Simulation Method. This method is a preferred approach for VaR computations, and is also often used in quantitative finance field. Monte-carlo method is the mightiest and perfect method for calculating the value of market risk. Here the Monte-Carlo method is used for estimating VaR and CVaR.

3 Discussion and Results

The topics discussed in this section are considered in two parts, first we are going to obtain the value of VaR and CVaR under minimal entropy martingale measure for exponential Levy process by Monte-Carlo simulation. Then, let us consider the propositions expressed in implied volatility under minimal entropy martingale measure in the preceding section.

3.1 Computing VaR and CVaR under MEMM for Levy Process

Here, we are going to obtain the value of VaR and CVaR under minimal entropy martingale measure for exponential Levy process by Monte-Carlo simulation. To do this, we first need to compute the value of price and returns. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t; 0 \geq t \geq T)$, and that a price process $S_t = S_0 e^{Z_t}$ of a stock is defined on this probability space, where Z_t is a Levy process. Now, we are going to obtain the value of option price, returns, implied volatility, value-at-risk and conditional value-at-risk by Monte-Carlo simulation. For this purpose, we have to calculate the formula (23) under minimal entropy martingale measure such as below:

$$E_{P^*}[F(X)] = E_P[\frac{dP^*}{dP} F(X)] \tag{29}$$

where

$$F(X) = (\frac{1}{M} (\sum_{i=1}^M S_{t_i} - K)^+) \tag{30}$$

and

$$\frac{dP^*}{dP} = e^{\theta^* \hat{Z}(t) - \hat{\Psi}(\theta^*) T} \tag{31}$$

In formula (29) can be seen kind of importance sampling that the main probability P changes to the probability P^*, with minimal relative entropy with respect to the P and the value of θ^* achieve from the equation (11). $\Psi(\theta)$ is characteristic function that stated in equation (4).

The exponential Levy process has two kinds of representation such that

$$S_t = S_0 e^{Z_t} = S_0 \mathcal{E}(\hat{Z})_t \tag{32}$$

Z_t and $\hat{Z}_t$ are named the compound return process and simplereturn process of S_t respectively. The MEMM is got as an Esscher-transformed martingale measure using the simple return process.

Here, it is reviewed the discrete time approximation of exponential Ley process. According to the two kinds of expression of S_t, obtained the following approximation formula. Assume that

$$S_{t_i} = S_0 e^{Z_{t_i}^{(n)}} \tag{33}$$

and

$$S_{t_i} = S_0 \mathcal{E}\left(\hat{Z}_{t_i}^{(n)}\right) \tag{34}$$

where $\mathcal{E}\left(\hat{Z}_{t_i}\right)$ is the discrete time stochastic exponential of $\hat{Z}_{t_i}^{(n)}$, and $\hat{Z}_{t_i}^{(n)}$ is stated using the bellow equation:

$$e^{Z_{t_i}^{(n)}} = \mathcal{E}\left(\hat{Z}_{t_i}^{(n)}\right) = \prod_{j=1}^{t_i}(1 + (\hat{Z}_{t_j}^{(n)} - \hat{Z}_{t_{j-1}}^{(n)})) \tag{35}$$

So we obtain

$$e^{\Delta Z_{t_i}^{(n)}} = e^{Z_{t_i}^{(n)} - Z_{t_{i-1}}^{(n)}} = \left(1 + (\hat{Z}_{t_i}^{(n)} - \hat{Z}_{t_{i-1}}^{(n)})\right) = 1 + \Delta \hat{Z}_{t_i}^{(n)} \tag{36}$$

The convergence of the process $Z^{(n)}(t)$ to $Z(t)$ can be easily proved when n goes to ∞.

$$\Delta \hat{Z}(t) \cong \hat{Z}(t) = \sum_{i=1}^{n} e^{Z\left(\frac{i}{n}T\right) - Z\left(\frac{i-1}{n}T\right)} - n \tag{37}$$

And suppose that the Levy measure of the variance gamma process Z_t is as follows:

$$v(dx) = C\left(\frac{1_{\{x<0\}}(x)e^{-c_1 x} + 1_{\{x>0\}}(x)e^{-c_2 x}}{|x|}\right)dx \tag{38}$$

Where there exist the constants C, c_1, c_2 so that $C > 0$, $c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$. Now, we will have,

$$\Phi(\theta^*) = b_0 + C\left(\int_0^\infty \frac{1}{x} e^{-c_1 x}\left((e^x - 1)e^{\theta^*(e^x - 1)} - 1_{x\in[0,1]}(x)\right)dx\right) + \tag{39}$$

$$C\left(\int_0^\infty \frac{1}{x} e^{-c_2 x}\left((e^x - 1)e^{\theta^*(e^x - 1)} + 1_{x\in[0,1]}(x)\right)dx\right) = r$$

$$\hat{\Psi}(\theta^*) = \mu\theta^* + \frac{\theta^*}{2}\sigma^2(1 + \theta^*) + \int_{\mathbb{R}\backslash\{0\}}[e^{\theta^*(e^x - 1)} - 1 - x\theta^* 1_{|x|\leq 1}(x)]v(dx) \tag{40}$$

In the following an algorithm for computing VaR using sample stock has been reviewed.

Algorithm 3.1.1: computing value-at-risk by Monte-Carlo method

The basic stages for computing value-at-risk using Monte-Carlo method:

1. Over time horizon T, time spacing Δt_i by dividing equally.

2. Generate N scenarios for the path of stock price $\Delta S^{(1)}, \dots, \Delta S^{(N)}$ over all of them Δt until reaching the end of T.

3. For using the Monte-Carlo method, repeat the previous step a large number to generate different paths.

4. For all of the paths, find the value of terminal returns.

After computing the returns and sort them, these must be counted and numbered. Then, the VaR is calculated for different values of α. Now, to find out exactly how much will loss on average in the worst case scenarios we have to compute CVaR values.

3.1.1 A Numerical Example

In this example, we first simulate a variance gamma process and then calculate option price and returns. Finally, we obtained VaR and CvaR using Monte-Carlo method. The chart of VaR and CVaR is plotted and compared for $\alpha = 5\%, 1\%, 0.1\%$. In the plotted chart can be seen decreasing VaR and CVaR by increasing the size of discretization grid. In this section, simulations are performed by MATLAB software and data are analyzed and plotted in Excel.

A variance gamma model is achieved by assessing a Brownian motion with a drift; at a random time given by a gamma process. The variance gamma model $(Y_t)_{t \in [0,T]}$ with parameters (σ, v, θ) is defined as:

$$Y_t = \theta \gamma_t + \sigma B_{\gamma_t} \tag{41}$$

where $(B_t)_{t \in [0,T]}$ is a standard Brownian motion and $(\gamma_t)_{t \in [0,T]}$ is a gamma process with unit mean rate and variance rate κ.

To describe the stock price, only a drift part has been added to the VG process. The price process is defined by an exponential Levy model where

$$Z_t = \gamma t + Y_t \tag{42}$$

The process $(Z_t)_{t \in [0,T]}$ is a Levy process.

An algorithm to simulate a variance gamma process is in the following. We have implemented an exponential variance gamma model using this algorithm and formula (42).

Algorithm 3.3.1. Simulating VG stock price (see [3])

VG parameters (σ, v, θ) ; time spacing $\Delta t_1, \ldots, \Delta t_N$ subjected to $\sum_{i=1}^{N} \Delta t_i = T$

Set $X_0 = 0$

1. Generate N independent gamma variable $(\Delta G_i \sim Gamma)$ with parameters $\frac{t_1}{v}, \frac{t_2 - t_1}{v}, \ldots, \frac{t_N - t_{N-1}}{v}$. Set $\Delta G_i = v \Delta G_i$.
2. Generate N i.i.d normal $Z_i \sim N(0,1)$
3. Return $\Delta X(t_i) = \theta \Delta G_i + \sigma \sqrt{\Delta G_i} Z_i$
4. For discretized trajectory set $X(t_i) = \sum_{i=1}^{N} \Delta X(t_i)$
5. Compute stock price: $S(t_i) = S(0)\exp(at + X(t_i))$

After simulating $\hat{Z}(t)$, we need to obtain θ^* from formula (55). These computations are performed using the Monte-Carlo and iteration methods, which can be found in [6].

It can be see that $\Phi(\theta)$ is a non-decreasing function of θ. Therefore, if $\Phi(\theta)$ is continuous and satisfies the following inequality:

$$\lim_{\gamma \to -\infty} \Phi(\theta) < r < \lim_{\gamma \to -\infty} \Phi(\theta) \tag{45}$$

Then the equation $\Phi(\theta) = r$ has a unique solution and the MEMM exists. By using the iteration method same as bellow, got an approximate solution for $\Phi(\gamma) = r$.

$$\theta_{n+1} = \theta_n - \frac{f(\theta_n) - r}{f'(\theta_n)} \tag{44}$$

The value of $\widehat{\Psi}(\theta^*)$ is also calculated using the Monte-Carlo method by Sobol sequence (see [20]).

To do this, we apply the above formula and replacing the Levy measure of the variance gamma.

$$\widehat{\Psi}(\theta^*) = \tag{45}$$
$$\{\mu\theta^* + \frac{\theta^*}{2}\sigma^2(1+\theta^*) + C\int_0^\infty \frac{1}{|x|}e^{-c_1 x}\left(e^{\theta^*(e^x-1)} - 1 - x\theta^* 1_{|x|\leq 1}(x)\right)(dx) +$$
$$C\int_0^\infty \frac{1}{|x|}e^{-c_2 x}\left(e^{\theta^*(e^x-1)} - 1 - x\theta^* 1_{|x|\leq 1}(x)\right)(dx)\}$$

By using the Monte-Carlo method, the next expression will be gained:

$$\widehat{\Psi}(\theta^*) = \tag{46}$$
$$\{\mu\theta^* + \frac{\theta^*}{2}\sigma^2(1+\theta^*) + CE_{c_1}\left[\frac{e^{\theta^*(e^x-1)} - 1 - x\theta^* 1_{|x|\leq 1}(x)}{c_1 x}\right] +$$
$$CE_{c_2}\left[\frac{e^{\theta^*(e^x-1)} - 1 - x\theta^* 1_{|x|\leq 1}(x)}{c_2 x}\right]\}$$

The value of θ^* for parameters ($\sigma = 1, v = 0.2, \theta = -0.01, \gamma = 0.1$) is equal to -0.5303 and the value of $\widehat{\Psi}(\theta^*)$ for the same parameters by Sobol sequence is equal to 0.03. Finally, after calculating the option prices and returns using the proposed algorithm, we obtain the VaR and CVaR. Fig. 1 represents the call option with respect to strike K by parameters ($\sigma = 1, v = 0.2, \theta = -0.01, \gamma = 0.1$). In Fig. 2 the implied volatility surface as a function of time to maturity and moneyness for the variance gamma process using equation (24) has been drawn. As shown in Fig. 2, the smile/smirk properties can be observed for the case $\tau \to 0$. Also, as time to maturity increases, the chart becomes more flat. This fact is visible in the proposition 3.2.2.

The VaR and the CVaR of the returns under the model are computed. Fig. 3 illustrate the variation of VaR and estimating VaR with respect to the size of discretization grid for different values of α. The value of VaR_α is stable from N=10000. Table 1 shows the same values for different N in percent. These charts are descending which is quite logical. Because with increasing discretization points accuracy increases. This fact is also evident in the table of variation of VaR_α. On the other hand, there is another matter in these charts that takes our attention. It's that as the alpha increases, the value-at-risk decreases. This is also economically true because the risk of more capital is less likely.
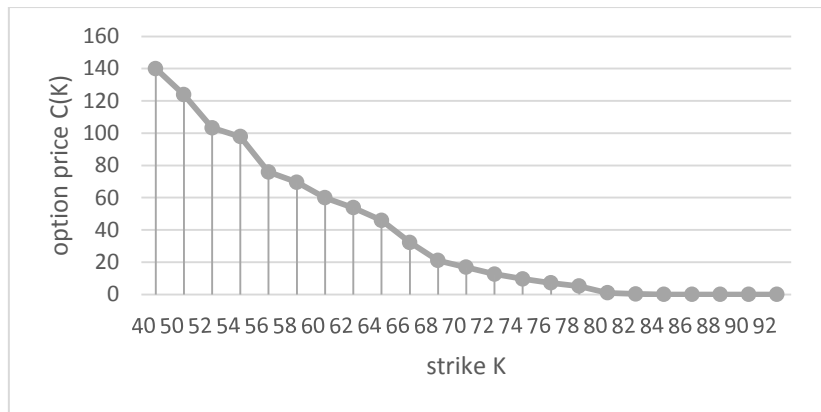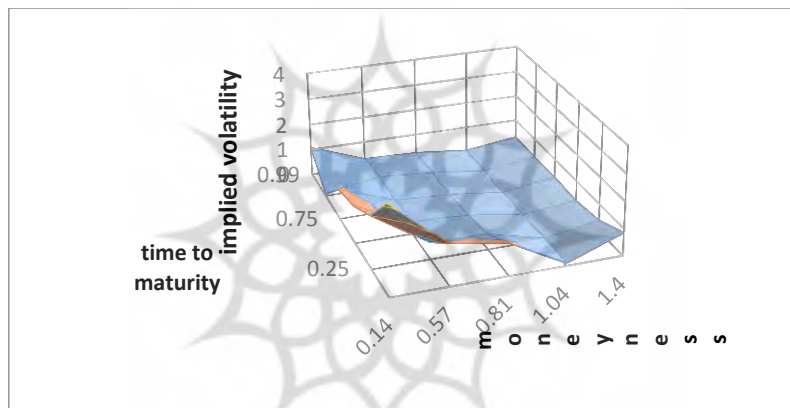
Fig 1: Option Price With Respect to Strike K

Fig 2: Implied Volatility Surface as a Function of Time to Maturity and Moneyness for The Variance Gamma Model
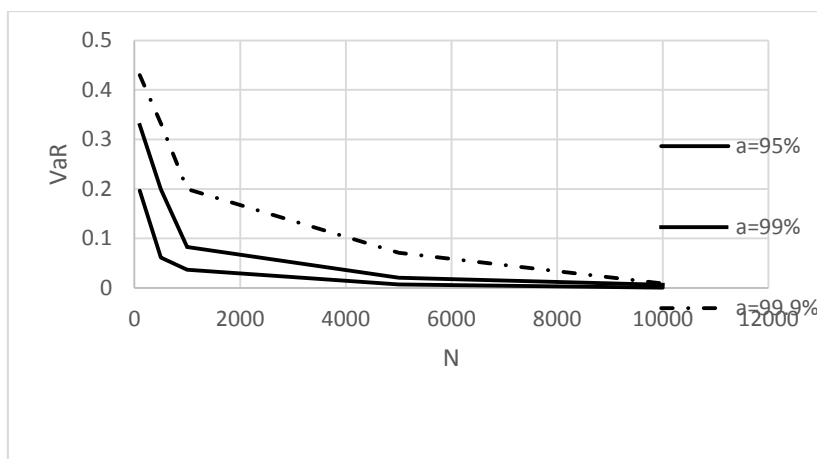
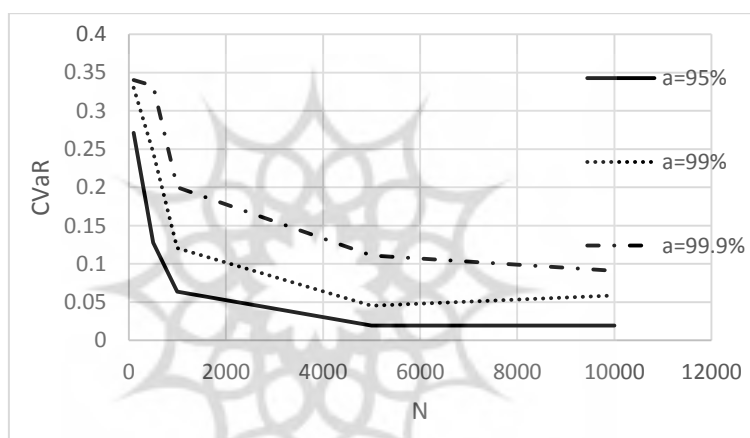**Fig 3**: Variation of VaR_α With Respect to N

Fig. 4: Variation of $CVaR_\alpha$ With Respect to N

Table 1: Variation of VaR_α by Increasing n

N	a=95%, $\alpha = 5\%$,	a=99%, $\alpha = 1\%$,	a=99.9%, $\alpha = 0.1\%$,
100	20%	33%	43%
500	6%	20%	33%
1000	4%	8%	20%
5000	1%	2%	7%
10000	0.04%	0.6%	1%

Table 2: variation of $CVaR_\alpha$ by increasing N

N	a=95%, $\alpha = 5\%$,	a=99%, $\alpha = 1\%$,	a=99.9%, $\alpha = 0.1\%$,
100	27%	33%	34%
500	13%	25%	33%
1000	6%	12%	20%
5000	2%	5%	11%
10000	2%	6%	9%

3.2 Asymptotic Implied Volatility under MEMM

In this section has been presented two propositions about asymptotic implied volatility under Esscher martingale measure for exponential Levy process. We attempt to investigate these two propositions under minimal entropy martingale measure. The behavior of implied volatility in exponential Levy models is very various from what is seen in stochastic volatility models with continuous-paths in short maturity. Comparing the option price asymptotic in the Black-Scholes model to those in the exponential Levy model is used for computing the short maturity asymptotic of implied volatility smile in exponential Levy models. In [19] has been shown two presented about short maturity asymptotic for at the money option and flattening of the smile far from maturity for Levy process under Esscher transform martingale measure. These two propositions have been shown in the following.

Proposition 3.2.1. (Short maturity asymptotic: ATM options) suppose that X be a Levy process without diffusion component and the corresponding Levy measure satisfying in the following condition

$$\int_{|x|\leq 1} |x|\nu(dx) < \infty \tag{47}$$

Then, the implied volatility $I(\tau, 0)$ in the state of at the money for the exponential Levy process $S_t = S_0 e^{X_t}$ falls as $\sqrt{\tau}$ for short maturities:

$$\lim_{\tau\to 0} \frac{I(\tau, 0)}{\sqrt{2\pi\tau}\max(\int (e^x - 1)^+ \nu(dx), \int (1 - e^x)^+ \nu(dx))} = 1 \tag{48}$$

The continuation of this proposition and its proof can be found in [19]. The next proposition has been first discussed in the [22] and has been implemented for exponential Levy model and stochastic volatility model. Then, this has been stated under Esscher transform martingale measure in the [19]. The representation is as follows.

Proposition 3.2.2. (Flattening of the smile far from maturity) suppose that X be a Levy process and the corresponding Levy measure satisfying in the following condition

$$\int_{|x|>1} x\nu(dx) < \infty \quad , \int_{|x|>1} xe^x \nu(dx) < \infty \tag{49}$$

Then the implied volatility $I(\tau, k)$ for the exponential Levy process $S_t = S_0 e^{X_t}$ satisfies

$$\lim_{\tau\to\infty} I^2(\tau, k) = 8\sup_\theta\{\frac{\sigma^2}{2}(\theta - \theta^2) - \int_{\mathbb{R}} (e^{\theta x} - \theta e^x - 1 + \theta)\nu(dx)\} \tag{50}$$

Here it is attempted to investigate the proposition 2.3.1 under MEMM. According to a remark in the [16] if the Levy measure satisfy in the following condition

$$\int_{|x|\leq 1} |x|\nu(dx) < \infty \tag{51}$$

the equation (11), which is the condition of existence for MEMM, is changed as follows:

$$b + \left(\frac{1}{2} + \gamma^*\right)\sigma^2 + \int_{-\infty}^{\infty}(e^x - 1)e^{\gamma^*(e^x-1)}\nu(dx) = 0 \tag{52}$$

and it's corresponding generating triplet (A^*, ν^*, b^*) under this probability as defined below :

$$A^* = \sigma^2 \tag{53}$$

$$\nu^*(dx) = e^{\gamma^*(e^x-1)}\nu(dx) \tag{54}$$

$$b^* = b + \gamma^*\sigma^2 \tag{55}$$

using the above equations, we will have the following statement

$$E[(1 - e^{X_\tau})^+] =$$
$$E\left[b\int_0^\tau e^{X_T}1_{X_T\leq 0}dt + \int_0^\tau\int_{\mathbb{R}} e^{\gamma^*(e^x-1)}\nu(dx)\{(1 - e^{X_\tau+x})^+ - (1 - e^{X_\tau})^+\}dt\right] \tag{55}$$

let X be a Levy process without diffusion component. Now, using L'Hopital's rule, $\lim_{\tau\to 0} E[e^{X_\tau}1_{X_\tau\leq 0}] = 1_{b\leq 0}(\frac{X_t}{t} \to b \ a.s. t \to 0)$ and dominated convergence the bellow relation achieved:

$$\lim_{\tau\to 0}\frac{1}{\tau}E[(1 - e^{X_\tau})^+] = b1_{b\leq 0} + \int_{\mathbb{R}} e^{\gamma^*(e^x-1)}\nu(dx)(1 - e^x)^+ \tag{56}$$

by martingale condition in (27), the above limit can be obtained as

$$\lim_{\tau\to 0}\frac{1}{\tau}E[(1 - e^{X_\tau})^+] = \max\left(\int e^{\gamma^*(e^x-1)}\nu(dx)(e^x - 1)^+, \int e^{\gamma^*(e^x-1)}\nu(dx)(1 - e^x)^+\right) \tag{57}$$

Comparing the above expression with the Black-Scholes ATM asymptotic stated as:

$$C_{BS}(\tau, k, \sigma) \sim \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}} \tag{58}$$

Then, the implied volatility $I(\tau, 0)$ in the state of at the money option for the exponential Levy process under MEMM is as follows:

$$\lim_{\tau\to 0}\frac{I(\tau, 0)}{\sqrt{2\pi\tau}\max(\int(e^x - 1)^+ e^{\gamma^*(e^x-1)}\nu(dx), \int(1 - e^x)^+ e^{\gamma^*(e^x-1)}\nu(dx))} = 1 \tag{59}$$

Let us now consider the theorem 2.3.2 under MEMM. Suppose that X be a Levy process and the corresponding Levy measure satisfies in the following condition

$$\int_{|x|>1} xe^{(1-e^x)}v(dx) < \infty \tag{60}$$

on the one hand, we know that the following condition is necessary for the existence MEMM:

$$\int_{|x|>1} (e^x - 1)e^{\gamma(1-e^x)}v(dx) < \infty \tag{61}$$

where the new probability $\mathbb{P}^*$ introduced by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} |_{\mathcal{F}_t} = e^{(e^{X_t}-1)} \tag{62}$$

let $\alpha^* = E^*[X_1]$ and defined as follows:

$$\alpha^* = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} \left(e^{(1-e^{X_t})}e^x - e^{(1-e^{X_t})}x - e^{(1-e^{X_t})}\right)v(dx) < 0 \tag{63}$$

the Cramer theorem is used to continue the proof.

Theorem (Cramer) 3.1: For an i.i.d sequence of random variables $\{X_i\}_{i\geq 1}$ with $E[X_i] = 0$ and all $x \geq 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} \ln P\left[\frac{S_n}{n} \geq x\right] = -\Gamma^*(x) \tag{64}$$

where the rate of convergence $\Gamma^*(x)$ is characterized by the Fenchel-Legender transform of the cumulant generating function Γ of X_1, defined as follows:

$$\Gamma^*(x) = sup_{\theta\in\mathbb{R}}[\theta x - \Gamma(\theta)] \in [0, \infty], x \in \mathbb{R} \tag{65}$$

using Cramer's theorem, the following equations are stablished:

$$\Gamma(\theta) = \log E^*[e^{-\theta(X_1-\alpha^*)}] \tag{66}$$

and

$$^*(\;^*) \;\; sup\{\frac{^2}{2}(\;\;0\;\;^2)0 \quad \int (e^{0\;x}e^{(10e^{0\;x})} \quad xe^{(10e^{0\;x})}0\,e^{(10e^{0\;x})})(0\;\;)\;(dx)$$

$$\int (\;e^x e^{(10e^x)}0 \quad xe^{(10e^x)}0\;\;e^{(10e^x)})\;(dx)\} \;\; sup\{\frac{^2}{2}(\tau 0\;\;^2)$$

$$\int (\;e^{(10e^{0\;x})})(e^{0\;x}\quad x\,01)\;(dx) \quad \int (\;e^{(10e^x)})(\;e^x 0\;\;x\,0\;\;)\;(dx)\}$$

$$sup\{\frac{^2}{2}(\;\;0\;\;^2)0 \quad \int (\tau e^{(10e^{\;x})})(e^{\lambda x}0\,Zx\,01)v(dx) \quad \int (\lambda e^{(10e^x)})(\theta e^x 0\,\varphi x\,0\,\tau),\;(dx)\}$$

according to the raised conditions in the beginning of the statement, the function $\Gamma^*(\alpha^*)$ are finite and continuous in the neighborhood of α^*. The answer of the above equations can be fined in the interval $\theta \in [0,1]$. Using Cramer's theorem can be seen:

$$\lim \frac{1}{\tau} \log(10 c(\tau, k)) \quad \sup_{[0,1]} \{\frac{\sigma^2}{2}(\theta^0 \quad \theta^2)$$
$$0 \quad \int (e^{(10e^x)})(e^{\theta x} \quad 0 \quad \theta x \quad 01) \quad (dx) \quad \int (e^{(10e^x)})(e^x \quad 0 \quad x \quad 0 \quad) \quad (dx)\} \tag{67}$$

The rest of the proof is like the statements for proposition 2.3.2 in [19]. Then the implied volatility $I(\tau, k)$ for the exponential Levy process satisfies:

$$\lim I^2(\tau, k) \quad 8 \sup \{\frac{\sigma^2}{2}(\theta^0 \quad \theta^2)$$
$$0 \quad \int (e^{(10e^x)})(e^{\theta x} \quad 0 \quad \theta x \quad 01) \quad (dx) \quad \int (e^{(10e^x)})(e^x \quad 0 \quad x \quad 0 \quad) \quad (dx)\} \tag{68}$$

4 Conclusion

In this paper, we pursued two goals that are based on two important and attractive concepts. These two purpose are considering and computing the implied volatility and risk measurement and those two important concepts are Levy process and minimal entropy martingale measure. That's here the price process of an underlying asset is an exponential Levy process and the price of an option is computed by minimal entropy martingale measure. The value of VaR and CVaR by Monte-Carlo simulation are obtained. For this purpose, we had to calculate option price and returns under MEMM and then we obtained risk measurement by proposed algorithm. This Martingale measure is used for the exponential type of the processes such as exponential Levy process. In fact, we can say MEMM is a kind of important sampling method where the probability measure with minimal relative entropy replaces the main probability. We used this to calculating the value of option price. Also, we obtained VaR and CvaR using Monte-Carlo method for exponential variance gamma process by calculate the value of option price and returns. For this example, after simulating the model, we considered the variation of risk measurements. The chart of VaR and CVaR is plotted and compared for $\alpha = 5\%, 1\%, 0.1\%$. In the plotted chart can be seen decreasing VaR and CVaR by increasing the size of discretization grid. Finally, we intended to develop two theorems for implied volatility under minimal entropy martingale measure by examining the conditions. These theorems present short maturity asymptotic for at-the-money option and flattening of the smile far from maturity for Levy process under MEMM.

Reference

[1]   Bakhshmohammadlou, M., Farnoosh, R., *Numerical Solution of Multidimensional Exponential Levy Equation by Block Pulse Function*, Advances in Mathematical Finance and Applications, 2020, **5**(2), P. 247-259. Doi: 10.22034/amfa.2020.1873599.1260

[2]   Carr, P., Geman, H., Madan, D. B., Yor, M., *The fine structure of asset returns: An empirical investigation,* The Journal of Business, 2002, **75**(2), P.305-332. Doi:10.1086/338705

[3] Cont, R., Tankov, P., *Financial Modeling with Jump Processes*, 2004, Chapman & Hall/CRC, Boca Raton. ISBN: 9781584884132

[4]   Delbaen, F., Grandits, P., Rheinländer, T., Samperi, D., Schweizer, M., Stricker, C., *Exponential hedging and entropic penalties*, Mathematical finance, 2002, **12**(2), P. 99-123. Doi: 10.1111/1467-9965.02001

[5] Fujiwara, T., Miyahara, Y., *The minimal entropy martingale measures for geometric Lévy processes*, Finance and Stochastics, 2003, **7**(4), P.509-531. Doi:10.1007/s007800200097

[6] Fujisaki, M., Zhang, D., *Evaluation of the MEMM, parameter estimation and option pricing for geometric Lévy processes,* Asia-Pacific Financial Markets, 2009, **16**(2), P.111-139. Doi: 10.1007/s10690-009-9089-1

[7] Glasserman, P., Heidelberger, P., Shahabuddin. P., *Importance sampling and stratification for value-at-risk,* IBM Thomas J. Watson Research Division,1999. (www.gsb.columbia.edu/faculty/pglasserman/Other/pg-var.pdf)

[8] Glasserman, P., Heidelberger, P., Shahabuddin, P., *Variance reduction techniques for estimating value-at-risk,* Management Science, 2000, **46**(10), P.1349-1364. ISBN:0-7803-6582-8

[9] Glasserman, P., Heidelberger, P., Shahabuddin, P., *Efficient Monte Carlo methods for value-at-risk*, IBM Thomas J. Watson Research Division, 2000.

[10] Hubalek, F., Sgarra, C., *Esscher transforms and the minimal entropy martingale measure for exponential Lévy models*, Quantitative finance, 2006, **6**(02), P.125-145. Doi: 10.1080/14697680600573099

[11] Izadikhah, M., *Improving the Banks Shareholder Long Term Values by Using Data Envelopment Analysis Model*, Advances in Mathematical Finance and Applications, 2018, **3**(2), P.27-41. Doi: 10.22034/amfa.2018.540829

[12] Jeanblanc, M., Klöppel, S., Miyahara, Y., *Minimal fq-martingale measures for exponential Lévy processes*, The Annals of Applied Probability, 2007, **17**(5/6), P.1615-1638. Doi: 10.1214/07-AAP439

[13] Jeanblanc, M., Yor, M., Chesney, M., *Mathematical methods for financial markets*, Springer Science & Business Media, 2009. ISBN-13: 978-1852333768

[14] Kind, A., *Pricing American-style options by simulation*, Financial markets and portfolio management, 2005, **19**(1), P. 109-116. Doi: 10.1007/s11408-005-2300-0

[15] Miyahara Y., Moriwaki N., *Volatility Smile/Smirk Properties of [GLP & MEMM] Models,* Discussion Papers in Economics, Nagoya City University, **405**, 2004, P.1-16.

[16] Miyahara, Y., *Option pricing in incomplete markets: Modeling based on geometric Lévy processes and minimal entropy martingale measures*, World Scientific, 2011, **3**. ISBN-13:978-1848163478

[17] Navidi, S., Rostamy-Malkhalifeh, M., Banihashemi, S., *Using MODEA and MODM with Different Risk Measures for Portfolio Optimization,* Advances in Mathematical Finance and Applications, 2020, **5**(1), P. 29-51. Doi: 10.22034/amfa.2019.1864620.1200

[18] Paziresh, M., Jafari, M., Feshari, M., *Confidence Interval for Solutions of the Black-Scholes Model,* Advances in Mathematical Finance and Applications, 2019, **4**(3), P. 49-58. Doi: 10.22034/amfa.2019.1869742.1231

[19] Tankov, P., *Financial modeling with Lévy processes,* 2010. HAL Id: cel-00665021

[20] Tahmasebi, M., Yari, G. H., *Minimal relative entropy for equivalent martingale measures by low-discrepancy sequence in Lévy process,* Stochastics, 2019, P. 1-18. Doi: 10.1080/17442508.2019.1642339

[21] Tehranchi, M. R., *Asymptotics of implied volatility far from maturity*, Journal of Applied Probability, 2009, **46**(3), P. 629-650. Doi: 10.1239/jap/1253279843

[22] Tehranchi, M. R., *Implied volatility: long maturity behavior*, Encyclopedia of Quantitative Finance, 2010. Doi: 10.1002/9780470061602.eqf08011

[23] Tilley, J. A.*, Valuing American options in a path simulation model*, Insurance Mathematics and Economics, 1995, **2**(16), P. 169. Doi:10.1016/0167-6687(95)91768-h