Tarikh-e Elm, 9(2011), pp. 1-19

Lost or Embedded Works of Kūhī

J. L. Berggren
Simon Fraser University, Canada
berggren@sfu.ca
(received: 05/07/2011-accepted: 20/09/2011)

Abstract

Abū Sahl Wījan (Bīzhan) ibn Rustam al-Kūhī the 10th century mathematician and astronomer was from Tabaristān (present Māzandarān) province of Iran. Apart from his rich legacy kept in several manuscripts and studied, edited and translated widely, we know that some of his works have been lost. In this article, Kūhī's lost mathematical works are traced through references and quotations in the works of other scholars, especially his contemporary al-Sijzī, and Kūhī himself.

Keywords: Abū Sahl Kūhī, lost works, al-Sijzī, Islamic mathematics

Introduction

Abū Sahl Wījan b. Rustam al-Kūhī[1] was a mathematician from Tabaristān (Ibn al-Nadim, I, p. 669) who flourished in the second half of the 10th century CE. His work was well-known among the mathematicians of his age working in the Buyid domains and he had as patrons at least three kings of the Buyid Dynasty: ‘Aḍud al-Dawlah, Sharaf al-Dawlah and Ṣamṣām al-Dawlah, whose combined reigns cover the period 962 – 998 CE. In the times immediately following him, Ibn al-Haytham and al-Bīrūnī knew of several of his works, and ‘Umar Khayyām cited him as one of the “distinguished mathematicians of ‘Irāq (Khayyām, p. 53).[2] In the first half of the fourteenth century a number of his works were studied by Muḥammad b. Sirtāq, and the 18th century scholar Muṣṭafā Ṣidqī not only copied several of his works but was sufficiently interested in his treatise on the volume of the paraboloid to make a shorter version of it.[3]

Kūhī had, however, influenced not only later geometers, but his contemporaries as well. Among them were some half-dozen other geometers whose works variously cite[4], complement and contrast with those of Kūhī. In view of all this we are fortunate in having available not only some thirty-two works by the great Iranian geometer, but letters that he wrote, and prefaces to some of his works, in which he makes a number of comments on the mathematics of his time.

One would think that his thirty-two known works would pretty-well cover his output. Yet, when Prof. Jan P. Hogendijk and I prepared a paper some years ago (Berggren and Hogendijk, cf. references) on fragments of Kūhī’s lost works in the writings of al-Sijzī, it struck both of us that only two of the ten continuous fragments of Kūhī’s works that al-Sijzī cites in his *Geometrical Annotations* are

---

1. This is simply one possible version of his name, based on a number of variants in the manuscripts. The most common reading is, in fact, Abū Sahl Wījan b. Rustam al-Qūhī. Wījan is the arabicized version of the Persian name Bīzhan which is still popular as a given name for men in Iran. Kūhī means someone who lives in or comes from mountain.
2. Khayyām’s reference to a mathematician from Tabaristān as a mathematician of ‘Iraq is explained by the fact that elsewhere in his *Algebra* (p. 83) Khayyām refers to him as being a member of a group of distinguished mathematicians in the court of ‘Aḍud al-Dawla in “The City of Peace”, i.e. Baghdad.
3. On Muṣṭafā Ṣidqī b. Ṣāliḥ see King, *Synchrony…*, vol. I, p. 306, note 1, as well as King, *Survey…*, p. 112.
4. To take only one example, Abū Naṣr b. ‘Irāq, in his *Al-Masā'il al-Handasiyya*, cites Kūhī’s *Establishing Points on Lines in the Ratio of Areas*.

from works whose texts survive today. The purpose of this paper is to see what one can learn about Kūhī from works of his that, although lost, are partially embedded in other works. We shall focus on his mathematical works, which are as follows:

THE WORKS

- *Establishing Points on Lines in the Ratio of Areas*
- *Answers to Questions Posed by Other Geometers*[1]
- *Discourse Embellishing Archimedes'* Lemmata
- *Filling a Lacuna in Book II of Archimedes'* Sphere and Cylinder[2]
- *On Centers of Gravity*
- *Division of the Sphere by Planes*
- *The Elements, According to the Model of the Book of Euclid*

To begin with evidence from Kūhī's works in al-Sijzī's writings we have ten substantial fragments from Sijzī's *Geometrical Annotations*. The first six of these ten are related to the following treatises of Apollonius: *The Conics*, *On Cutting-off a Ratio*, *On Plane Loci*, and *The Determinate Section*.[3]

The first two fragments contain problems which are theorems from *The Conics* specialized to circles. The first is simply *Conics*, III, 53 stated for special case of a circle (rather than any central conic). Otherwise, it differs from Apollonius's work only in the interchange of two of the figure's six letters. It seems likely that this piece was meant as instructional material for al-Sijzī and suggests that the relationship between Kūhī and al-Sijzī might have been that of teacher and student.

1. This is the title I have chosen to refer to a lost work, containing an anthology of problems and their solutions. It is clear from items 10, 1-6 below that Kūhī wrote some such work as this, although no later writer refers to it by title.
2. *Al- Fihrist* (Ibn al-Nadim, p. 669) cites this title as "Additions to the Second Book of Archimedes."
3. Of these works the first is extant in Greek, the first two in Arabic, and the last two known only from Pappus's summary in Book VII of his *Mathematical Collection*.

In the second fragment Kūhī considers [Fig. 1] two tangents at points *E* and *Z* of a circle that intersect at point *A*. He proves that if a straight line through *A* intersects the circle at points *T* and H and chord *EZ* at *I* then *HA*:*AT*=*HI*:*IT*.

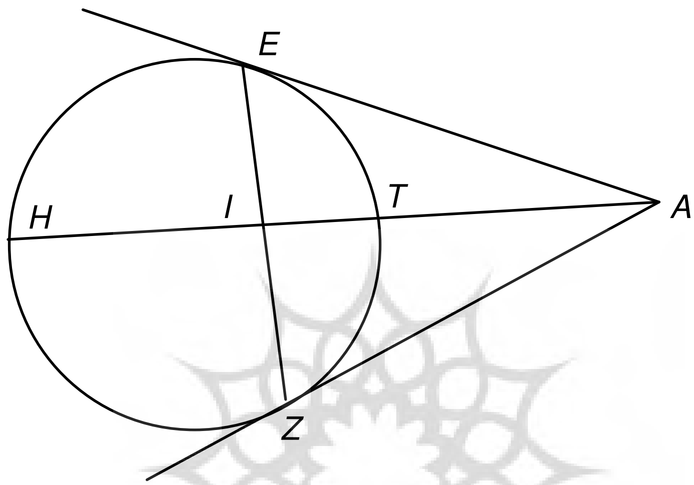

Fig. 1

Again, this is a special case of *Conics*, III, 37, where it is stated for any section of a cone – including the two branches of a hyperbola. Kūhī's proof is more complicated than that of Apollonius, but – unlike that of Apollonius – it does not need previous theorems in *Conics*, III. Moreover, it can also be used to prove the specialization of *Conics*, III, 38 to a circle. Finally, it is related to the first problem in Kūhī's *Two Geometrical Questions* (Berggren and Van Brummelen, 2001), which is a special case of a theorem in Apollonius's *On Plane Loci*.

In the next fragment, 3, from al-Sijzī's treatise, [Fig. 2] Kūhī shows that if side *AB* of triangle *ABG* is greater than side *BG* then the inscribed square with a side on *AB* is less than that with a side on *BG*.

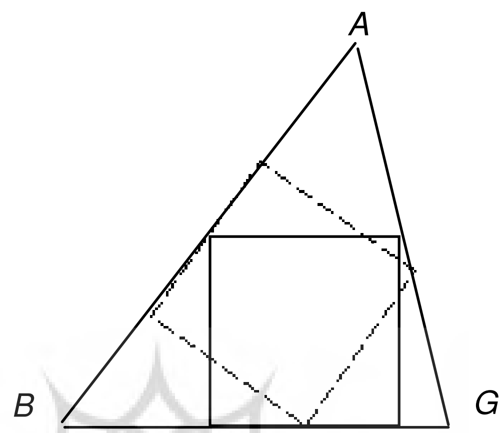

Fig. 2

The study of figures of one type inscribed in those of another was a lively tradition in 10th-century geometry To take only the case dealt with here, Kūhī's contemporary, Abu'al-Wafā' al-Būzjānī, gives two constructions for squares inscribed in triangles in his *Essentials of Geometry for Craftsmen*. Kūhī wrote on a more difficult case of the same problem, inscribing equilateral pentagons in squares (Hogendijk, 1985).

Fragment 4 [Fig. 3] gives an analysis of the following general problem from Apollonius's *Cutting-Off a Ratio*:

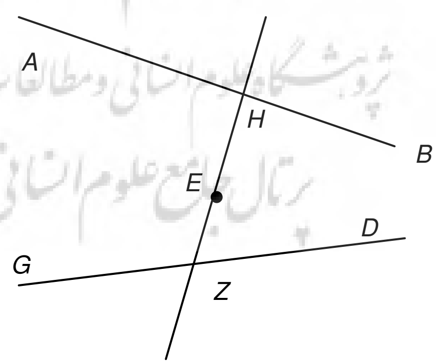

Fig. 3

Given straight lines *AB* and *GD*, and points *A* and *G* on *AB* and *GD* respectively, let *E* be a third given point somewhere in the plane. Draw through *E* a straight line *HEZ* so that *AH*:*ZG* is equal to a given ratio.

In fragment 5 Kūhī poses the problem [Fig. 4] of inscribing a triangle of given area in a semicircle.

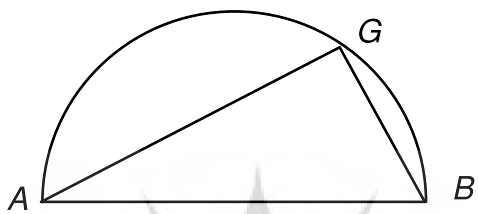

Fig. 4

Solving this very easy problem does not depend on the curve being a semicircle. That Kūhī would have realized this is clear from the last problem of his treatise *On the Ratio of the Segments of a Single Line that Cuts Three Lines* (Berggren and Van Brummelen, 2000). One suspects, then, that the problem in fragment 5 was also instructional material for al-Sijzī.

Fragment 6 [Fig. 5] contains two problems, which probably came from a single treatise by Kūhī.

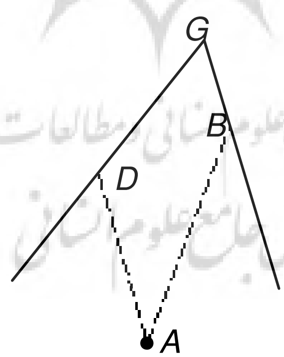

Fig. 5

Problem 6, 1 requires that one draw from a given point, *A*, to two given lines, *DG* and *GB*, two lines, *AD* and *AB*, that contain a given angle and whose product is equal to a given rectangle. In its solution Kūhī uses an idea that he also employs in the solution of a problem he solves in his correspondence with the famous Buyid vizier, Abū Isḥāq al-Ṣābī (Berggren, 1983). Problem 6, 1 relates to the first locus in Apollonius's *Plane Loci*, and it provides the first evidence of knowledge of Book I of that work in Arabic.[1]

In problem 6, 2 Kūhī replaces the requirement that the product of *AD* and *AB* is a given area by the requirement that they have a given ratio to each other. The two problems are variations, in reverse order, of the first two problems of Kūhī's *Drawing Two Lines from a Given Point at a Known Angle by Analysis* (Berggren and Van Brummelen, 2001). In this latter treatise Kūhī solves the problem of drawing two lines from *A* to a given straight line or circle so that the other stated conditions are satisfied. Unlike this latter treatise, however, where we find only analyses, we find in fragment 6 the syntheses as well.

In Fragment 7 [Fig. 6] Kūhī poses the problem of a triangle *ABG* divided into two parts by the line segment, *BD*, through vertex *B*.

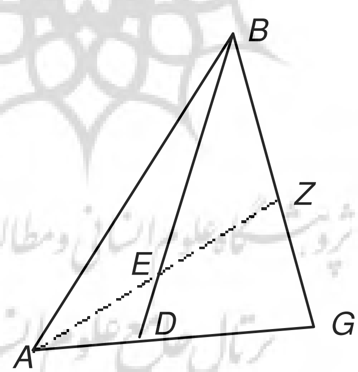

Fig. 6

1. A discussion of the loci provided in Apollonius's book may be conveniently found in Heath, vol. II, pp. 185-192. There is no evidence that Pappus' Book VII was known in medieval Islam, so this latter work was not a source for Kūhī.

The task is to construct a line segment *AZ*, through vertex *A*, cutting *BG* at *Z* and *BD* at *E*, so that the ratio of triangle *ADE* to *BZE* is given. Al-Sijzī's lengthy solution to this problem does not concern us here.

In fragment 8 [Fig. 7] Kūhī poses and solves a problem in which one is given two line segments, *AE* and *GZ*, and a point *H*.

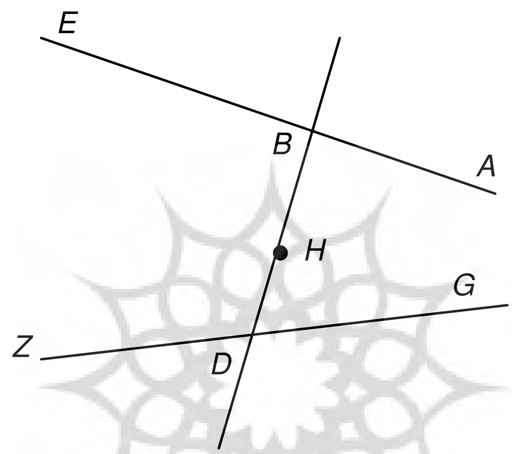

Fig. 7

The problem is to draw a line *DHB* cutting *AE* at *B* and *GZ* at *D* so that the ratio of rectangle *GD* by *AB* to the rectangle *ZD* by *EB* is equal to a given ratio. Kūhī uses the method of analysis to reduce the problem to a problem that he had solved in his lost treatise *Establishing Points on Lines in the Ratios of Areas*, one that is a variant of the problem from *Establishing* that Kūhī refers to in his treatise on the astrolabe (Berggren, 1994). This treatise, *Establishing*, is, as we shall make clear later, closely related to Apollonius's work, *The Determinate Section*.

Fragment 9 [Fig. 8] provides the diorismos, or limitation, for fragment 5. It concerns the right triangle formed by dropping a perpendicular, *GD*, from a point, *G*, on a semicircle of diameter *AB*, to that diameter.

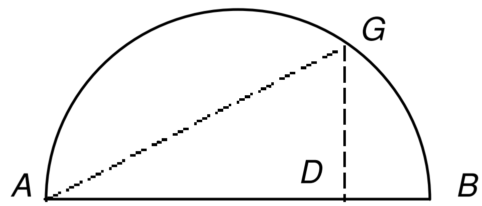

Fig. 8

It asks for a proof of the fact that the point *G* producing the largest such right triangle *ADG* is the point dividing the circumference, *ABD*, so that arc *BG* is one third of the circumference of the semicircle. The real work, of course, is not proving the result but discovering it. Since Kūhī told al-Sijzī what to prove, this is further evidence of a pedagogical purpose in some of his dealings with al-Sijzī.

The lengthy fragment 10 is headed "Answer by Wījan, known as Abū Sahl al-Kūhī, to the Geometrical Questions."

The first four problems in this fragment remind one of Euclid's *Division of Figures*, and all four use the ancient technique of application of areas from Book II of Euclid's *Elements*. The first two problems of this fragment (10, 1 & 2) concern cutting a triangle by a straight line and can be reduced to a later problem in the group, 10, 6. Such problems as these were popular in 10th century 'Iraq and al-Sijzī solves other problems of this sort in his *Geometrical Annotations*.

Problem 10, 5 asks for the construction of a line that cuts two given circles in chords of given lengths.

Problem 10, 6 is a verging construction, asking that one construct a line through a given point that cuts off from a given angle a triangle of given area. Here again we have the same two cases of applications of areas as in 10, 1-3.

Problems 10, 1-6, then, are probably taken from a treatise of Kūhī's, written in response to a list of questions that had been posed to him. Both Ibrāhīm b. Sinān and al-Sijzī also wrote works of this sort. The treatise as a whole resembles Kūhī's extant treatise, *Two Geometrical Questions* (Berggren and Van Brummelen, 1999-2000), which might also be part of the same treatise.

We turn now to a lost treatise of Kūhī that is cited in two of his major extant works, and is referred to by a number of other mathematicians. This is his

Establishing Points on Lines in the Ratios of Areas, which is related to Apollonius's *Determinate Section*.

In Book II of his work *On the Construction*[1] *of the Astrolabe with Proofs*, Kūhī simply refers to the work and says that two of its theorems justify his argument. The context is in one of many problems in the treatise, which is concerned with the problem of reconstructing an astrolabe given some information about the location of some points or curves on it.

Then, later in Book II, he quotes the two theorems, and gives an analysis of each. Here he uses a shorter form of the title, *Establishing Points on Lines*. In the first theorem, a straight line segment, *AB*, is divided into three segments by known points *G* and *D*, and the problem is to find a third point, *E*, on segment, *GD*, so that the areas formed by two rectangles created from the four segments having *E* as an endpoint are in a given ratio. In the second theorem, *AB* is divided into two segments by a single point, *G*, and the problem is to find a point, *E*, on the segment, *GB*, so that the areas formed by the two rectangles created from the four segments created by this division are in a given ratio.

As we mentioned above, this work is very closely related to Apollonius' work *The Determinate Section* whose principal problem is the following[2]:

Fig. 9

One is given [Fig. 9] four points on a line – say *A*, *B*, *G* and *D* – and it is required to find a point *P* on that line so that the ratio of the rectangle whose sides have lengths *AP* and *GP* to that whose sides have lengths *BP* and *DP* is equal to a given ratio. This is, in fact, a general form of the first of Kūhī's problems, and – as Heath remarks – a complete discussion by Apollonius would involve a discussion of *diorismoi*, the limits of possibility of a solution, including a discussion of the relative position of the point-pairs, *A*, *G* and *B*, *D*. The Greek text of this work is

1. The Arabic word used is '*iḥdāth*.'
2. See T. L. Heath, *History of Greek Mathematics*, vol. II, 180-81.

lost, but its contents are summarized in Book VII of Pappus' *Mathematical Collection*.

This work of Apollonius was translated into Arabic, and Abū Isḥāq al-Ṣābī, in his correspondence with Kūhī, reminds him that he had promised to send him "the remaining theorems of the second book of Apollonius' treatise *The Determinate Section*" (Berggren, 1983). One wonders, in fact, if Kūhī's *Establishing Points on Lines* was not a re-working of Apollonius's treatise.

There is a second book in which Kūhī also refers to his *Establishing Points on Lines*, and that is his *Perfect Compass* (Rashed, cf. references)[1]. In this work Kūhī applies the second theorem referred to above (in the case when the point *G* is on an extension of *AB*, beyond *A*, say) in constructing a hyperbola satisfying certain given conditions.[2] The form in which he cites the title is exactly as in section II, 3 of his treatise on the astrolabe. However, he does not, in *On the Perfect Compass*, cite the result he is using, much less analyze or prove it.

His more casual use of the results in the work *On the Perfect Compass* suggests that the work was written after that on the astrolabe and he believed that his readers would have been familiar with the results he needed from his full citation of them in his work on the astrolabe.

Further, in regard to Kūhī's *Establishing Points*, the historian F. Woepcke in a note to his translation of *On the Perfect Compass*, calls attention to another reference to Kūhī's *Establishing Points on Lines* in a treatise on the perfect compass by one Muḥammad b. al-Ḥusayn[3] written for the library of Ṣalāḥ al-Dīn (Saladin) sometime between 1187/583 and 1193/588. Muḥammad tells us that he read of Kūhī's treatise *Establishing* in al-Bīrūnī's *Complete Exposition of All Possible Ways of Constructing an Astrolabe*, but that he, himself, was unable to find this treatise.

1. There is an earlier translation of this work (together with text and commentary) by F. Woepcke, available in his "Trois Traités Arabes sur le Compas Parfait" reprinted in vol. 2 of *Études sur les Mathématiques Arabo-Islamiques* (ed. F. Sezgin), Frankfurt, 1986, pp. 561-734.
2. Woepcke, II, p. 650; Arabic text, p. 721, line -2.
3. Woepcke published this treatise on the basis of a manuscript (Leiden MS. 1076) that also contained, besides the works of Muḥammad and Kūhī on the perfect compass, a work by Kūhī's contemporary, al-Sijzī.

And finally, in regard to *Establishing Points*, Abū Naṣr b. 'Irāq, who was a contemporary of Kūhī, hints at the context for this work when he writes to al-Bīrūnī, in response to questions from the latter, as follows:[1] "The questions that you collected in your letter have arrived, and you mentioned that three of them are contained in the book of Kūhī *On the Perfect Compass*, and that in it he relied on his book *Establishing Points on the Lines According to the Ratios of the Areas*. But since you did not have access to that book [by Kūhī], you asked some geometer of our time about it, and he answered, using [in his answer] the properties of conic sections. But you were not pleased by this [solution] because they (i.e., al-Bīrūnī's questions) are preliminaries for finding conic sections, which [preliminaries] are prior in the [logical] order. So you asked me their construction by means of the geometrical elements[2] and the methods of the craftsmen, etc." Since the three questions all concern constructing conic sections, perhaps this was the original context for Kūhī's work.

Of Kūhī's work, *Division of a Sphere by Planes,* Rosenfeld and Ihsanoğlu say it is found in Tehran Sipahsalar 693, but Sezgin (V, p. 320, #20) says that the Tehran manuscript is, rather, a treatise borrowing from Kūhī's treatise of that name, by one 'Alī Muḥammad Isfahānī (ca 1800 AD). Iṣfahānī writes in his treatise that before him only Abū Sahl Kūhī had written a book on that subject (Ghorbani, p. 429).

We now turn to Kūhī's "*Filling a Lacuna in Book II of Archimedes'* " (Berggren, 1994), our source for our knowledge of this treatise being the part (or, quite possibly, all of it) in Naṣīr al-Dīn Ṭusi's commentary on Book II of Archimedes' *On the Sphere and Cylinder*. In it, Kūhī solves the problem suggested to him by those Archimedes solves in that work, namely the problem of constructing a segment of a given sphere whose volume is equal to that of a given segment of one sphere and whose surface is equal to that of a given segment of a

1. I thank Prof. J. P. Hogendijk for kindly sending me the information quoted here (in his translation) about the contents of Abū Naṣr's treatise.
2. It is unclear whether this refers to Euclid's *Elements* or simply to basic plane geometry, without the theory of conic sections.

second sphere. Ṭūsī makes it clear that he is quoting the title verbatim; however, the title in *al-Fihrist* is *Additions to the Second Treatise of Archimedes*.

Further evidence for Kūhī's close study of Archimedes' works comes from Kūhī's *Embellishing the* Lemmata, which Sezgin (V, p. 133) lists as "*Discourse on Embellishing Archimedes'* Lemmata". About this work al-Nasawī tells us that it contains a better proof of the fifth theorem of the *Lemmata* than that given by Archimedes,[1] in which two semicircles are constructed on segments *AB* and *BG* of the diameter *AG* of a given semicircle *ADG*.

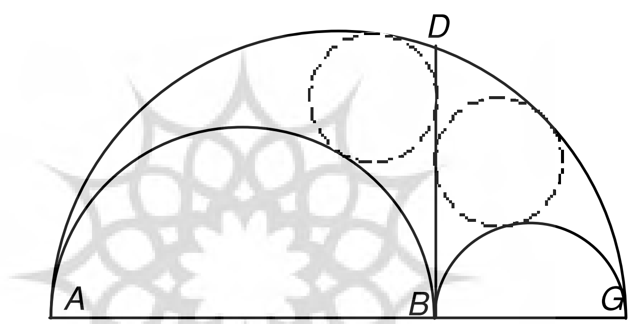

Fig. 10

The theorem in the *Lemmata* states that if *BD* is the perpendicular to *AG* from *B*, then the two circles shown in dashed lines in Fig. 10 are equal. .

It turns out that Theorem 5 of Kūhī's work concerns the famous figure of the *arbelos*, and generalizes the theorem of Archimedes. Indeed, Heath (p. 307) writes of this interesting proposition that "As pointed out by… Alkauhi [sic] this proposition (5), concerning the equality of two circles inscribed in the arbelos, may be stated more generally".

Hogendijk has published Kūhī's extension, to the case when the two semicircles are not tangent and Kūhī's result consists of two parts: (1) If [Fig. 11] the interior semicircles intersect at *W*, drop a perpendicular, *WE*, from *W* to the base, *AD*, and extend it upward to meet the bounding semicircle at *D*. Then the

1. Sezgin (V, p. 133) refers the reader for details to Steinschneider, *Die mittleren Bücher der Araber und ihre Bearbeiter*, Zeit. für Math. u. Physik 10/1865/480.

inscribed circles corresponding to the ones in *Lemmata* Prop. 5 are equal; (2) If the interior semicircles are disjoint and *E* is the point on the segment between them from which the tangents to the two semicircles are equal, then a result corresponding to that in (1) holds in this case as well (Hogendijk, 2008).

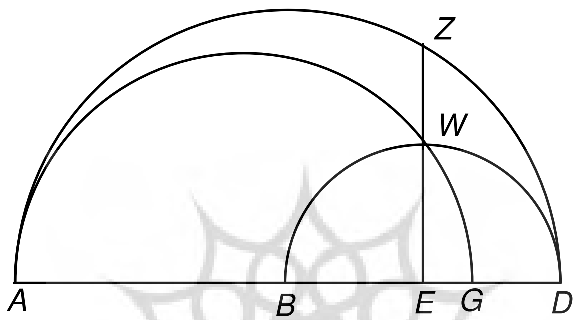

Fig. 11

Because Naṣīr al-Dīn al-Ṭūsī included Kūhī's results in his commentary to his edition of the *Lemmata*, European mathematicians of the Renaissance became acquainted with them.[1] In the seventeenth century the famous English mathematician, Isaac Barrow, included them in his Latin translation of Archimedes' works. It later appeared also in Dutch mathematical texts (Hogendijk, 2008).

1. It appears, then, that these two results from al-Kūhī's study of Archimedes' *Lemmata* are the only pieces of his work known to European mathematicians in the Renaissance and early modern times.

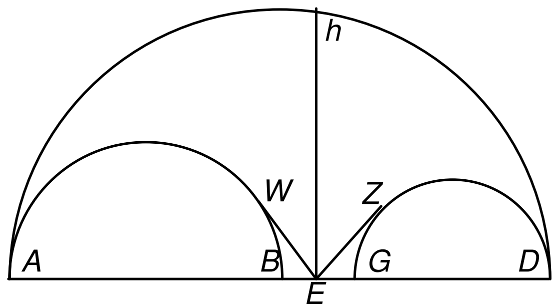

Fig. 12

This work strengthens the impression created by Kūhī's work on Book II of *On the Sphere and Cylinder* namely that he believed there was still gold in the streams that Archimedes had explored, and he systematically looked for it.

Sezgin (p. 320) states that Kūhī wrote a treatise on the fact that the ratio of the circumference of a circle to its diameter is $3^1/_9$ but cites no manuscript, only a refutation by Ibn Ṣalāḥ Hamadānī (d. 548 AH/1153 AD) also known as Ibn al-Sarī (Sesiano, cf. references). The title of the Arabic treatise by Ibn Ṣalāḥ is "Treatise on refutation of Abū Sahl Kūhī's treatise on [showing] that the ratio of the diameter [of circle] to [its] circumference is [equal to] the ratio of 1 to $3^1/_9$" (Ghorbani, p. 36).

Among Kūhī's works that al-*Fihrist* mentions is *The Elements According to the Model of the Book of Euclid, with what Issued from It*, but no copy of this work is known. The title suggests a reworking of the *Elements* with some additional material, and we do find extant studies of Books I and II of Euclid's work by Kūhī that certainly fit this description. For example, Kūhī's *Additions to the Elements,* II (De Young, cf. references), presenting a radical re-working of the first ten propositions of Book II, which De Young found interpolated in three manuscripts of that work, along with 17 new propositions that Kūhī added to the book. Subsequently, Berggren and Van Brummelen (2003) published a translation of a short Arabic extract that an anonymous medieval mathematician described as being "from what Kūhī added at the end of Book II for the study of Books II and III of the *Conics*." The first four propositions are also found among the 17 new propositions in the work De Young published. But, the fifth proposition, which concludes what is labeled as a quote from the works of Kūhī, is new. With the exception of Proposition 4, all five propositions do indeed fill gaps in Apollonius' arguments in Books II and III, and Proposition 4 is needed for the proof of Proposition 5.

More important than the 'new' proposition, however, is the information that the context for Kūhī's work on Book II of the *Elements* was aimed at easing the study of the *Conics*. Following our publication of this short treatise, Van Brummelen and the present author published (Berggren and Van Brummelen, 2005) Kūhī's drastic re-working of Book I. Neither of Kūhī's extensive works on *Elements* I and II contains any prefatory material. Like the *Elements* itself, they begin with the mathematics and contain nothing but mathematics. This is at least consistent with their being extracts from the larger work mentioned in *al-Fihrist*.

In his correspondence with al-Ṣābī, Kūhī informs us about a work he was writing at the time of the correspondence, *On Centers of Gravity*. He tells us that he had all but finished 6 books of the work, that he was going to do a 7th which would be the longest and best, then he would do three or four on liquid and non-liquid bodies, and then write the introductory chapter. Although the number of books in the final work is uncertain, there can be no doubt that he did write such a work, for Sesiano (pp. 281-82) cites the encyclopedist, Shams al-Dīn Muḥammad al-Anṣārī al-Damashqī (d. 727 AH/1327 AD), in evidence of this, and he also reproduces a passage of Ibn Ṣalāḥ's in which the latter says he found only two of the four books Kūhī said he had written on centers of gravity.

In addition, 'Abd al-Raḥmān al-Khāzinī presents a summary of theorems he attributes to Kūhī and Ibn al-Haytham on centers of gravity in his *Balance of Wisdom*, which he completed in 1121. Although he does not say so one presumes that al-Khāzinī's summary from Kūhī was taken from his *On Centers of Gravity*. Unfortunately, al-Khāzinī does not identify which of the nine sections in his Book I are by Kūhī and which are by Ibn al-Haytham. However, Bancel (2001) attempts to separate out the part that was due to Kūhī from that due to Ibn al-Haytham, the basis for deciding being the belief that Kūhī would have been more interested in a geometrical (over a physical) approach to the subject and would have focused on the determination of centers of gravity. On the other hand, the assumption is, Ibn al-Haytham would have been more likely to take a physical approach. Following on these suppositions Bancel assigns the first five sections to Kūhī and the last four to Ibn al-Haytham.

In any case, we know from Kūhī's correspondence with al-Ṣābī that his *On Centers of Gravity* contained demonstrations of the centers of gravities of two plane figures and three solid figures, since he explicitly says so and gives the correct results for these five figures.[1] In addition, he mentions in his treatise on the volume of a paraboloid that he found the center of gravity of a segment of an ellipsoid – something he does not mention in the correspondence. It thus appears from these fragments that Kūhī was the most original thinker in the Archimedean tradition of centers of gravity in the interval between the Syracusan mathematician and the Italian, Commandino, in the 15th century.

In summary, it seems that the evidence from Kūhī's lost works and those quoted by others suggest the following conclusions:

1. Kūhī was more involved in the production of instructional material than one had thought. The word 'instructional" is used here in the broad sense, to include not only material designed specifically for teaching purposes (as, for example, some of the material produced for al-Sijzī) but also reworking and connecting the ancient classics in a matter that would have benefited those studying them. Here one thinks of his *Elements on the Model of Euclid*, with the additions to *Elements* II, intended to support the study of early books of the *Conics*. The line between this type of activity and simply reworking a classic for an intellectual challenge becomes blurred at times, and so one should also mention here Kūhī's production of *Establishing Points on Lines* as being possibly a reworking of Apollonius' *The Determinate Section*.

2. Over 40 years ago (Anbouba, cf. references) called attention to the importance of correspondence between mathematicians in the Buyid era, and al-Sijzī's extensive quotation from "Answers by Wījan to the Geometrical

1. He also gives a result for a semicircle, which is incorrect. The pattern for the other figures suggest the result, and as a result of it he showed that pi is equal to $3^1/_9$!

Questions” provides further evidence that Kūhī was stimulated by interchange with his colleagues.

3. Kūhī was also stimulated by his study of classics, not only to solve old questions in a new way (e.g. constructing a heptagon and trisecting an angle), but also to frame new questions, as we know from his successful extension of results in Archimedes’ *On the Sphere and Cylinder,* II and his partially successful continuation of Archimedes’ work on centers of gravity.

In summary, the fragments of Kūhī’s works advance his image as a creative geometer, very much in the classical tradition, who thought it important not only to extend that tradition but to shape and re-form it in a way that would ease its study for future generations of students.

References:

Anbouba, A., “Qadiyyatun handasiyyatun wa muhandisūna fi’l-qarn al-rābi‘ al-hijrī tasbī‘ al-dā’irat,” *Journal for the History of Arabic Sci*ence, vol. 1, no. 2 (Nov. 1977), pp. 384-352 (A brief version of this paper may be found in the same author’s “Construction de l’heptagon regulier par les Arabes au 4e siècle”, *Journal for the History of Arabic Sci*ence, vol. 2, no. 2 (Nov. 1978), pp. 264-269).

Bancel, F., “Les centres de gravité d’Abū Sahl al-Quhi”, *Arabic Sciences and Philosophy* vol. 11 (2011), pp 45-78.

Berggren, J. L., “The Correspondence of Abū Sahl al-Kūhī and Abū Isḥāq al-Ṣābī: A translation with commentaries”, *Journal for History of Arabic Science*, vol. 7, 1983, nos. 1 & 2, pp. 39-123.

———, “Abū Sahl al-Kūhī's Treatise on the Construction of the Astrolabe with Proof: Text, Translation and Commentary”, *Physis*, vol. XXXI, Nuova Serie Fasc. 1 (1994), pp. 142-252.

———, “Al-Kūhī's Filling a Lacuna in Book II of Archimedes' *On the Sphere and Cylinder* in the Version of Naṣīr al-Dīn al-Ṭūsī”, *Centaurus*, vol. 38 (1996), pp. 140-207.

Berggren, J. L. and Hogendijk, J. P., “The Fragments of Abū Sahl al-Kūhī’s Lost Geometrical Works in the Writings of al-Sijzī”, *Studies in the History of the Exact Sciences in Honor of David Pingree*, ed. by C. Burnett et al., Leiden, 2004.

Berggren, J. L. and Van Brummelen, G., “Abū Sahl al-Kūhī on Two Geometrical Questions”, *Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften*, vol. 13 (1999-2000), pp. 165-187.

———, “Abū Sahl al-Kūhī's ‘On the Ratio of the Segments of a Single Line That Falls on Three Lines”, *Suhayl,* vol. I (2000), pp. 1-56.

———, “Abū Sahl al-Kūhī's ‘On Drawing two lines from a given point to a given line by the method of analysis”, *Suhayl*, vol. 2 (2001), pp. 161-198.

———, “From Euclid to Apollonius: Al-Kūhī's Lemmas to *The Conics*”, *Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften*, vol. 15 (2003), pp. 165-174.

———, “Al-Kūhī's Revision of Book I of Euclid's *Elements*” *Historia Mathematica*, vol. 32 (2005), no. 4, pp. 426-452.

De Young, G., “Abū Sahl’s Additions to Book II of Euclid’s *Elements*”, *Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften*, vol. 7 (1991-2), pp. 73-135.

Ghorbani, A., *Zendegīnāme-ye Rīyāżīdānān-e Dowre-ye Eslāmī* (“A Biography of the Islamic Period Mathematicians”), 2nd ed. Tehran, 1989.

Heath, T. L., *The Works of Archimedes*, New York, (n.d.).

Hogendijk, J. P., “Al-Kūhī's construction of an equilateral pentagon in a given square”, *Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften*, vol 1 (1985), pp. 100-144.

———, “Two beautiful geometrical theorems by Abū Sahl Kūhī in a 17th century Dutch translation”, *Tarikh-e Elm: Iranian Journal for the History of Science*, vol. 6 (2008), pp. 1-36.

Ibn al-Nadim, Abū’l-Faraj, *Al-Fihrist*, translated by Bayard Dodge (2 vols.), New York and London, 1970.

Khayyām, ‘Umar, *Algebra*, ed. and translated by A. Djebbar and R. Rashed in *L’Oeuvre Algébrique d’al-Khayyam*, Aleppo, 1981.

King, D. A. *In Synchrony with the Heavens*, vol. II, Leiden: Brill, 2005.

———, *A Survey of the Scientific Manuscripts in the Egyptian National Library*, (No. 5 in the series “Catalogs” of the American Research Center in Egypt) Winona Lake, IN: Eisenbrauns, 2005.

Rashed, R., “Al-Qūhī and al-Sijzī on the perfect compass and the continuous drawing of conic sections”, *Arabic Sciences and Philosophy*, vol. 13 (2003), pp. 9-43.

Sesiano, J., “Note sur trois théorèms de Mécanique d’al-Qūhī et leur conséquence”, *Centaurus*, vol. 22 (1979), no. 4, pp. 281-297.

Sezgin, F., *Geschichte des Arabischen Schrifttums*, vol. V, Leiden, 1974.