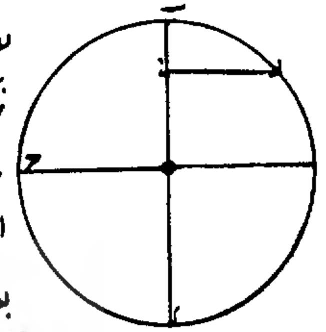
من رسالة له بسم الله الرحمن الرحيم وبه نستعين شرح عسيره لابي الفتح عمر بن ابرهيم الخيامي

نريد ان نقسم ربع دائرة آب من دائرة آبجد بقسمين على نقطة مثل ز ونخرج عمود زح

على قطر بج فيكون نسبة آه الى زح كنسبة هح الى حب وه مركز الدائرة وآه نصف القطر



نصف القطر فانا اذا انا فعلنا خرج بنا التحليل الى امر معلوم ثم

نركب على تلك الصفة فنفرض ان ربع آب ومركزها ه ونخرج آد

ونقطعها على زوايا قائمة ونخرج عمود زح فيكون نسبة آه

الى زح كنسبة هح الى حب ونخرج عمودي ي ز ط ونتم سطح طد

بمان جعلنا خط هم مثل آه فلان نسبة آه الى زح كنسبة ح

لانا ح دهم مثل آه يكون نسبة هم الى زح ضرب هم في حب مساويا لضرب زح في ح كما

بين اقليدس في تو من الاصول وضرب هم في حب مثل سطح كد وضرب زح في ح مثل سطح

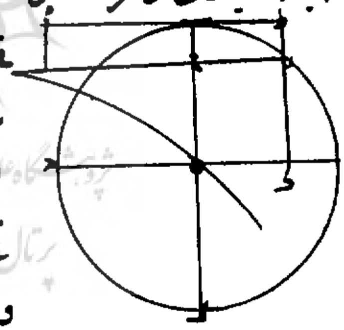
ح ى فيكون سطح كد مساويا لسطح ح ى ونجعل سطح ح ط

مشتركا فيكون سطح طه مساويا لسطح طد فان عملنا قطعا زائدا

لا يلقيا خطا طد طم ويمر على نقطة ح كما بين المونيوس

في تح من المقالة الاولى من كتاب المخروطات والشكل ح

وه من المقالة الثانية من هذا الكتاب اذ هذا العمل تم تهيك



الاشكال المشتركات ذلك القطع الزائد يمر على نقطة ز لانه كما بين من عكس الشكل الثامن من

المقالة الثانية من كتاب المخروطات ونقطة معلومة الوضع وخط تم معلوم الوضع والقدر الا

ان نقطة ز عند التركيب غير معلومة الوضع لانها لو كانت معلومة الوضع لكانت نقطة معلومة

الوضع لان خط ز معلوم القدر فيكون خط ح معلوم القدر ولكان الشكل معلوما وكذلك

not used in obtaining R.

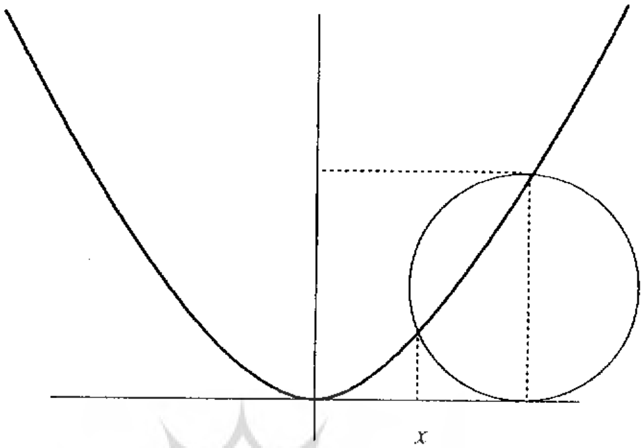


Fig. 3

Suppose the problem is solved and R is the desired point. We draw ER. The tangent line at R intersects EB at T. Then in the right triangle ERT we have

$$ER^2 = (EH)(ET) \tag{12}$$

and

$$RH^2 = (EH)(HT). \tag{13}$$

Let $EH = x, ER = 1, ET = t$. Then from (12) and (13) we obtain

$$1 = xt \tag{14}$$

and

$$RH = \sqrt{x(t-x)}. \tag{15}$$

Substituting (14) and (15) in (11), we get

$$\frac{1}{\sqrt{x(t-x)}} = \frac{x}{1-x}. \tag{16}$$

So from (16) we obtain

$$x^4 - 2x + 1 = 0 \tag{17}$$

We observe that 1 is a root of (17), but instead of factoring we treat it with Khayyām's technique.

We choose $y = x^2$. The with (17) we have

$$\begin{cases} y^2 - 2x + 1 = 0 \\ x^2 - y = 0. \end{cases} \tag{18}$$

From this set of equations we get the circle

$$(x-1)^2 + \left(y - \frac{1}{2}\right) = \frac{1}{4}. \tag{19}$$

Thuse the circle of center $(1, \frac{1}{2})$ and radius $\frac{1}{2}$ intersect the parabola $y = x^2$ at two points (Fig. 3). Note that since $x < 1$, the root $x = 1$ is

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Khayyām's Problem

An interesting problem had started 'Omar Khayyām on employing conic sections in solving cubic and quartic equations. So it is proper studying it.

We want to divide the one fourth AB of the circle $ABCD$ by a point R into two parts such that if RH is drawn perpendicular to the diameter BD, we obtain

$$\frac{AE}{RH} = \frac{EH}{HB}, \tag{11}$$

where E is the center of the circle (Fig. 2).

Khayyām's solution is quite involved with a long discussion of cubic equations,[3] we shall give a simpler solution.

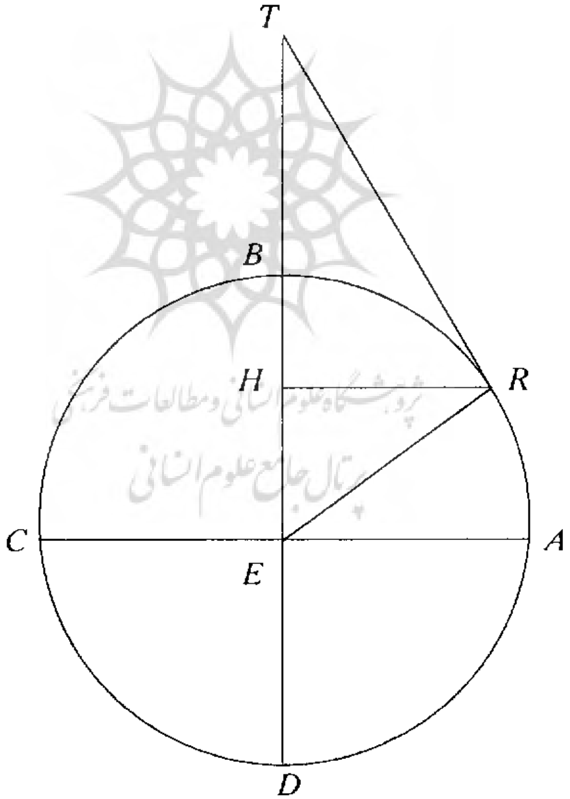

Fig. 2

3. Amir-Moéz, A.R. "A Paper of 'Omar Khayyām", *Scripta Mathematica*, vol. 26, no. 4, 1961, pp. 323-337.

The advantage of (8) is that parabola $y = x^2$ can be drawn accurately on a sheet of scaled paper. Then a circle of center

$$\left(\frac{-B}{2}, \frac{1-A}{2}\right) \tag{9}$$

and radius

$$\frac{\sqrt{A^2 + B^2 - 4AC - 2A + 1}}{2} \tag{10}$$

can be drawn on a sheet of transparent paper. We superimpose the circle on the parabola and read the roots on the x-axis (Fig. 1).

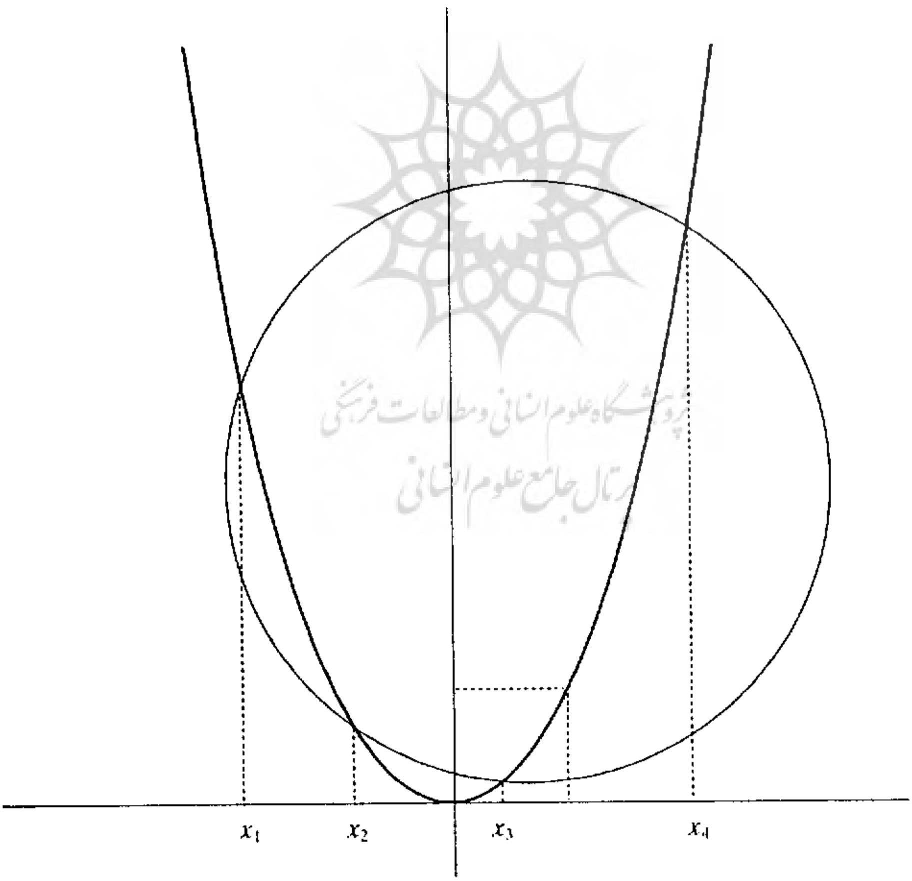

Fig. 1

It is clear that the axes of these parabolas are perpendicular to one another. Now if we add these equations we get

$$x^2 + y^2 - 4px - 4qy + 4ap + 4bq = 0 \tag{2}$$

which is a circle with center $(2p,\ 2q)$. This proves the theorem. Here the complex points of intersection have also been considered. Khayyām proved this theorem synthetically. We shall leave that to the reader as an exercise.

Solution of cubic equations

Any third degree equation can be written as

$$x^3 + lx^2 + mx + n = 0. \tag{3}$$

If we discuss the solution of a fourth degree equation such as

$$z^4 + az^3 + bz^2 + cz + d = 0, \tag{4}$$

then (3) will be a special case of (4). So, we consider

$$x^4 + lx^3 + mx^2 + nx = 0. \tag{5}$$

Then we ignore the root $x = 0$, and we get the roots of (3).

Now let us proceed with the solution of (4). If we choose the change of variable $z = x - (a/4)$, the equation (4) chauges to the form

$$x^4 + Ax^2 + Bx + C = 0. \tag{6}$$

We choose $y = x^2$. The getting the roots of (6) is the same as solving the system of equations

$$\begin{cases} x^2 = y \\ y^2 + Ay + Bx + C = 0 \end{cases} \tag{7}$$

for x.

It is easily seen that the equations of (7) are the equations of two parabolas whose axes are perpendicular to one another. The solution of (7) is obtained by the system

$$\begin{cases} x^2 = y \\ x^2 + y^2 + (A-1)y + Bx + C = 0. \end{cases} \tag{8}$$

Khayyām, Hashtroudī, and Quartic Equations

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'Omar Khayyām's solution of cubic equations which employs the parabola $y = x^2$ and circles, is applied to construction of some geometric configurations.

'Omar Khayyām (1044-1123), a Persian Mathematician, had used conic sections in construction of the roots of cubic equations.[1] An extension of his work which is due to the late. Dr. M. Hashtroudi ancient professor of mathematics at Tehran University,[2] uses only circles and parabolas. This way a simplification of Khayyām's work will be presented.

Theorem: *The four points of intersection of two parabolas whose axes are perpendicular are on a circle.*

Proof : For convenience, without loss of generality, we choose the parabolas

$$y^2 = 4p(x - a), \quad x^2 = 4q(y - b). \tag{1}$$

1. Woepcke, F. *L'Algèbre d'Omar Al-Khayyāmī*, publiée, traduite et accompagnée d'extraits manuscrits inédits, Paris, 1851.

2. Amir-Moéz, A.R. "Khayyām's Solution of Cubic Equations", *Mathematics Magazine*, vol. 35, no. 5, 1962, pp. 285-286.

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